On Measuring the Natural Rate of Interest⋆

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Abstract

This paper provides estimates of the natural rate of interest from Holston, Laubach and Williams' (2017) structural model which correct the implementation of Median Unbiased Estimation (MUE) in Stage 2 of their framework. The proposed correction is quantitatively important. It yields substantially smaller point estimates of the signal-to-noise ratio parameter λ_z which determines the size of the downward trend of '*other factor*' z_t in the natural rate. For US data, the point estimate of λ_z shrinks from 0.040 to 0.013 and is statistically highly insignificant. For data on the Euro Area, the UK and Canada, the λ_z point estimates are zero. These results show that the effect of '*other factor*' z_t on the natural rate is considerably smaller than originally estimated by Holston *et al.* (2017).

Keywords: Natural rate of interest, Median Unbiased Estimation, Kalman Filter, misspecified econometric models, correction to Stage 2 model.
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1. Introduction

Holston, Laubach and Williams' (2017) (simply HLW's henceforth) estimates of the natural rate of interest have become the benchmark rate — not only for policy makers at central banks — but also for asset managers and finance professionals concerned with long-term investment decisions.

This paper provides estimates of the natural rate of interest from HLW's structural model that take into account the correction to Median Unbiased Estimation (MUE) in Stage 2 of their model described in detail in Buncic (2020). The Stage 2 model in Holston *et al.* (2017) estimates the size of the signal-to-noise ratio parameter λ_z, which determines the severity of the downward trend of *'other factor'* z_t in the natural rate. This Stage 2 model is misspecified in Holston *et al.* (2017). Moreover, HLW's implementation of the dummy variable structural break regressions that produce the F statistics which are inverted to obtain the MUE of λ_z are modified from their original format proposed by Stock and Watson (1998). The combination of the misspecification of the Stage 2 model and the modification of the structural break regressions leads to spuriously large estimates of λ_z and exaggerated downward trends in the estimates of *'other factor'* z_t, and ultimately, the natural rate of interest.

The correction to HLW's Stage 2 model and MUE implementation that I propose is quantitatively important. For the US, the MUE point estimate of λ_z shrinks from 0.040 to 0.013, and is highly insignificant statistically. For the Euro Area, the UK, and Canada, the MUE point estimates are exactly zero. These zero point estimates of λ_z result in (filtered and smoothed) estimates of *'other factor'* z_t that are essentially flat lines centered at zero (or just below zero for the Euro Area). This suggests that the natural rate is solely determined by trend growth in this model and data. For the US, the downward trend in *'other factor'* z_t is substantially reduced, yielding point estimates that are approximately 100 basis points larger at the end of the sample period than from HLW's implementation.

The remainder of the paper is organized as follows. In Section 2, Holston *et al.*'s (2017) structural model of the natural rate of interest is briefly described, followed by a concise review of how MUE is implemented in Stock and Watson (1998). This section also shows how MUE on the misspecified Stage 2 model, as implemented in HLW, cannot recover the signal-to-noise ratio of interest $\lambda_z = \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$, before describing how the structural break testing regression is implemented in Stock and Watson (1998), and contrasting it to HLW's implementation. The empirical results of the correct Stage 2 MUE implementation, along with the results from HLW are reported in Section 3. The study is concluded in Section 4.

Accompanying Matlab and R code that replicates the results that are presented here — together with filtered and smoothed estimates of the natural rate, trend growth, *'other factor'* z_t, and the output gap — are provided on the author's website at: http://www.danielbuncic.com.

2. Holston *et al.*'s (2017) structural model and MUE in Stage 2

This section begins by outlining Holston *et al.*'s (2017) structural model of the natural rate of interest and proceeds by reviewing how Median Unbiased Estimation (MUE) is implemented in Stock and Watson (1998), before describing HLW's implementation of MUE in Stage 2 of their procedure. The correctly specified Stage 2 model and HLW's misspecified formulation of the Stage 2 model are then contrasted. I also show in this section that HLW's misspecified Stage 2 model cannot recover the signal-to-noise ratio of interest $\lambda_z = \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$. Lastly, HLW's implementation of the structural break dummy variable regression is compared to Stock and Watson's (1998) original implementation.

2.1. Model set-up in Holston *et al.* (2017)

Holston *et al.*'s (2017) structural model of the natural rate takes the following form:

$$\text{Output:} \quad y_t = y_t^* + \tilde{y}_t \tag{1a}$$

$$\text{Inflation:} \quad b_\pi(L)\pi_t = b_y \tilde{y}_{t-1} + \varepsilon_t^\pi \tag{1b}$$

$$\text{Output gap:} \quad a_y(L)\tilde{y}_t = a_r(L)\left[r_t - 4g_t - z_t\right] + \varepsilon_t^{\tilde{y}} \tag{1c}$$

$$\text{Output trend:} \quad y_t^* = y_{t-1}^* + g_{t-1} + \varepsilon_t^{y^*} \tag{1d}$$

$$\text{Trend growth:} \quad g_{t-1} = g_{t-2} + \varepsilon_{t-1}^g \tag{1e}$$

$$\text{Other factor:} \quad z_{t-1} = z_{t-2} + \varepsilon_{t-1}^z, \tag{1f}$$

where the terms $b_\pi(L) = (1 - b_\pi L - (1 - b_\pi)(L^2 + L^3 + L^4))$, $a_y(L) = (1 - a_{y,1}L - a_{y,2}L^2)$, and $a_r(L) = \frac{a_r}{2}(L + L^2)$ are lag polynomials that capture the dynamics in inflation π_t, the output gap $\tilde{y}_t$, and the real rate cycle $\tilde{r}_t = (r_t - 4g_t - z_t)$, and L is the lag operator. Output, denoted by y_t, is constructed as 100 times the log of real GDP, π_t is annualized quarter-on-quarter PCE inflation, and the real interest rate r_t is computed as $r_t = i_t - \pi_t^e$, where i_t is the nominal interest (federal funds) rate and expected inflation is approximated by $\pi_t^e = \frac{1}{4}\sum_{i=0}^{3} \pi_{t-i}$ in Holston *et al.* (2017). The natural rate of interest r_t^* is defined as the sum of annualized trend growth $4g_t$ and *'other factor'* z_t, both of which follow first order integrated processes ($I(1)$ henceforth). The error terms ε_t^ℓ in (1) are assumed to be *i.i.d.*, mutually uncorrelated, and with time-invariant standard deviations denoted by $\sigma_\ell, \forall \ell = \{\pi, \tilde{y}, y^*, g, z\}$.

Holston *et al.* (2017) argue that due to '*pile-up at zero*' problems with Maximum Likelihood Estimation (MLE) of the standard deviations of the innovations to trend growth g_t and *'other factor'* z_t, ML estimates of σ_g and σ_z are likely to be biased towards zero. Because of this, they estimate σ_g and σ_z indirectly by employing Stock and Watson's (1998) MUE of the signal-to-noise ratios $\lambda_g = \frac{\sigma_g}{\sigma_{y^*}}$ and $\lambda_z = \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$ in two preliminary stages, which are referred to as Stage 1 and Stage 2 in their three stage estimation procedure. In the section that follows, I briefly describe how MUE in Stock and Watson (1998) is implemented. Since σ_g in (1) can be estimated directly by MLE in the Stage 2 and Stage 3 models *without* any indication of '*pile-up* *at zero*' problems materializing, and since the focus of this paper is on the misspecification in the Stage 2 model, I describe MUE as it is implemented in the Stage 2 model only to conserve space and avoid unnecessary repetition.[1]

2.2. Median Unbiased Estimation in the Stage 2 model

Stock and Watson (1998) introduced MUE in the context of US trend growth per capita estimation in a local level model of the form:

$$GY_t = \beta_t + u_t \tag{2a}$$
$$\Delta\beta_t = (\lambda/T)\eta_t \tag{2b}$$
$$a(L)u_t = \varepsilon_t, \tag{2c}$$

where η_t and ε_t are two *i.i.d.* and mutually uncorrelated disturbance terms. They showed via simulation that when the standard deviation of $\Delta\beta_t$ ($\sigma_{\Delta\beta}$) is '*very small*', MLE leads to higher '*pile-up at zero*' frequencies of the estimate $\hat{\sigma}_{\Delta\beta}$ than MUE, in particular, when the initial condition of the state vector β_t is also estimated. Stock and Watson (1998) derived the MUE parameter of interest λ to be"*... T times the ratio of the long-run standard deviation of* $\Delta\beta_t$ *to the long run standard deviation of* u_t." (Stock and Watson (1998), p. 351, right column, top of the page), leading to the following λ_z and 'signal-to-noise ratio' relation (HLW use $\lambda_z = \frac{\lambda}{T}$):

$$\lambda_z = \frac{\lambda}{T} = \frac{\bar{\sigma}(\Delta\beta_t)}{\bar{\sigma}(u_t)} = \frac{\sigma_{\Delta\beta}}{\sigma_\varepsilon/a(1)}, \tag{3}$$

where $a(L)$ is an AR(4) lag polynomial and $\bar{\sigma}(\cdot)$ denotes the long-run standard deviation. Equation (3) gives a mapping between λ_z and the 'signal-to-noise ratio' $\frac{\bar{\sigma}(\Delta\beta_t)}{\bar{\sigma}(u_t)}$ in the local level model in (2). The Median Unbiased estimate of λ (or λ_z) is obtained by 'inverting' a structural break test statistic using the look-up values in Table 3 on page 354 in Stock and Watson (1998), which are based on structural break tests that test for a break in the unconditional mean of GY_t.[2]

In HLW's Stage 2 model, the 'signal-to-noise ratio' relation $\lambda_z = \frac{a_r\sigma_z}{\sigma_{\tilde{y}}}$ is used in the estimation of the full model in (1) (see page S64). To appreciate algebraically why $\lambda_z = \frac{a_r\sigma_z}{\sigma_{\tilde{y}}}$ in the Stage 2 model of HLW, assume for now that the latent cycle and trend growth variables $\tilde{y}_t$ and g_t, as well as all the parameters $a_{y,1}$, $a_{y,2}$ and a_r in (1) are known or observed. The objective is to obtain an estimate of the standard deviation of the increment to the latent process z_t (σ_z) by employing MUE. For that purpose, one needs to formulate a local level model

[1] See Section 4.1 in Buncic (2020) on how MUE is implemented in the Stage 1 model.

[2] Note that these look-up values were computed via simulation. How exactly Stock and Watson (1998) implement the structural break test is described in Section 2.4 below.

involving z_t which is analogous to the trend growth specification in (2) taking the form:

$$\overbrace{a_y(L)\tilde{y}_t - a_r(L)[r_t - 4g_t]}^{\text{analogue to } GY_t \text{ in (2a)}} = \overbrace{-a_r(L)z_t}^{\text{analogue to } \beta_t \text{ in (2a)}} + \varepsilon_t^{\tilde{y}} \tag{4a}$$

$$\underbrace{-a_r(L)\Delta z_t}_{\substack{\text{analogue to} \\ \Delta\beta_t \text{ in (2b)}}} = \underbrace{-a_r(L)\varepsilon_t^z}_{\substack{\text{analogue to} \\ (\lambda/T)\eta_t \text{ in (2b)}}}, \tag{4b}$$

where $-a_r(L)z_t$ is the 'time-varying mean' process captured by β_t in (2). The signal-to-noise ratio corresponding to (3) for the local level model in (4) is then:

$$\lambda_z = \frac{\lambda}{T} = \frac{\bar{\sigma}(\Delta\beta_t)}{\bar{\sigma}(\varepsilon_t^{\tilde{y}})} = \frac{\bar{\sigma}(-a_r(L)\Delta z_t)}{\sigma_{\tilde{y}}} = \frac{a_r(1)\sigma_z}{\sigma_{\tilde{y}}} = \frac{a_r\sigma_z}{\sigma_{\tilde{y}}}, \tag{5}$$

due to $a_r(1) = \frac{a_r}{2}(1+1^2) = a_r$ and $\bar{\sigma}(\varepsilon_t^{\tilde{y}}) = \sigma_{\tilde{y}}$ since $\varepsilon_t^{\tilde{y}}$ is assumed to be uncorrelated. The last term in (5) gives HLW's Stage 2 relation $\lambda_z = \frac{a_r\sigma_z}{\sigma_{\tilde{y}}}$. Once λ_z is estimated from Stock and Watson's (1998) structural break testing procedure using the look-up values in Table 3 of their paper to convert the structural break test statistic into the MUE of λ_z, σ_z is replaced by $\frac{\hat{\lambda}_z\sigma_{\tilde{y}}}{a_r}$ in the full model's likelihood function, and $\hat{\lambda}_z$ is held fixed in the estimation of the remaining parameters of the model.

2.3. HLW misspecified and the correctly specified Stage 2 models

In the derivation of the algebraic relationship between λ_z and $\frac{a_r\sigma_z}{\sigma_{\tilde{y}}}$ it was assumed that the latent cycle and trend growth variables $\tilde{y}_t$ and g_t, as well as all the parameters $a_{y,1}$, $a_{y,2}$ and a_r in (1) are known. In practice, however, these will need to be replaced by estimates obtained from a 'Stage 2' model for the implementation of the structural break tests in the construction of MUE to be feasible. This Stage 2 model should simply be defined as the full model in (1), but without *'other factor'* z_t in the output gap relation in (1c), and with the equation for z_t in (1f) removed from the model. This 'correctly specified' Stage 2 model, shown in the left column block in (6) below, is consistent with the full model's output gap definition in (1c) and yields HLW's signal-to-noise ratio $\lambda_z = \frac{a_r\sigma_z}{\sigma_{\tilde{y}}}$ as demonstrated algebraically in (4) and (5), and required for the estimation of the full model.

Holston *et al.* (2017), nevertheless, 'unnecessarily misspecify' the Stage 2 output gap equation by adding an intercept term and allowing for only one lag in trend growth in the model.[3] Moreover, the parameter on the lagged trend growth term is not restricted to be

[3] I use the term 'unnecessarily misspecified' here because the correct Stage 2 model can be easily obtained by simply deleting the last two row and column entries of the **F** and **H** matrices of their Stage 3 state-space model representation, while leaving the rest of the measurement equation unchanged. Their misspecified Stage 2 model instead requires also a modification of the $\mathbf{x}_t$ and **A** matrices. See the Appendix in Buncic (2020), specifically, pages A-3 to A-9, and/or the *'Documentation of R Code and Data for "Measuring the Natural Rate of Interest: International Trends and Determinants"'* pdf file which is included in HLW_Code.zip that contains the R-Code for exact details. The code is available from:

$-4a_r$ any more, but is instead estimated freely as a_g. To contrast their misspecified Stage 2 model from the correct one, I show their Stage 2 model in the right column block under the heading 'HLW misspecified' in (6) below.[4]

$$
\begin{array}{rlrl}
\multicolumn{2}{c}{\textbf{Correctly specified}} & \multicolumn{2}{c}{\textbf{HLW misspecified}} \\
y_t &= y_t^* + \tilde{y}_t & y_t &= y_t^* + \tilde{y}_t \qquad \text{(6a)} \\
b_\pi(L)\pi_t &= b_y \tilde{y}_{t-1} + \varepsilon_t^\pi & b_\pi(L)\pi_t &= b_y \tilde{y}_{t-1} + \varepsilon_t^\pi \qquad \text{(6b)} \\
a_y(L)\tilde{y}_t &= a_r(L)\left[r_t - 4g_t\right] + \mathring{\varepsilon}_t^{\tilde{y}} & a_y(L)\tilde{y}_t &= a_0 + a_r(L)\, r_t + a_g g_{t-1} + \mathring{\varepsilon}_t^{\tilde{y}}, \qquad \text{(6c)} \\
y_t^* &= y_{t-1}^* + g_{t-1} + \varepsilon_t^{y^*} & y_t^* &= y_{t-1}^* + g_{t-2} + \mathring{\varepsilon}_t^{y^*} \qquad \text{(6d)} \\
g_{t-1} &= g_{t-2} + \varepsilon_{t-1}^g & g_{t-1} &= g_{t-2} + \varepsilon_{t-1}^g \qquad \text{(6e)}
\end{array}
$$

The two error terms $\mathring{\varepsilon}_t^{\tilde{y}}$ in (6c) corresponding to the correctly and misspecified Stage 2 models are, respectively:

Correctly specified

$$\mathring{\varepsilon}_t^{\tilde{y}} = -a_r(L)z_t + \varepsilon_t^{\tilde{y}}$$

HLW misspecified

$$
\begin{aligned}
\mathring{\varepsilon}_t^{\tilde{y}} &= \underbrace{-a_r(L)4g_t - a_r(L)z_t + \varepsilon_t^{\tilde{y}}}_{\text{missing true model part}} - \underbrace{(a_0 + a_g g_{t-1})}_{\text{added Stage 2 part}} && \text{(7a)} \\
&= \underbrace{-a_r(L)z_t + \varepsilon_t^{\tilde{y}}}_{\substack{\text{required terms from} \\ \text{true model}}} - \underbrace{\left[a_0 + a_g g_{t-1} + a_r(L)4g_t\right]}_{\text{unnecessary terms}}, && \text{(7b)} \\
&= -a_r(L)z_t + \underbrace{\varepsilon_t^{\tilde{y}} - \left[a_0 + \tfrac{(a_g + 4a_r)}{2}(g_{t-1} + g_{t-2}) + \tfrac{a_g}{2}\varepsilon_{t-1}^g\right]}_{\mathring{\nu}_t^{\tilde{y}}}, && \text{(7c)} \\
&= -a_r(L)z_t + \mathring{\nu}_t^{\tilde{y}}. && \text{(7d)}
\end{aligned}
$$

To see the effect this 'unnecessary misspecification' in the Stage 2 model has on the λ_z and signal-to-noise ratio relation, one can again go through the same algebraic steps as in equations (4) and (5) above. That is, assume that $\tilde{y}_t$ and g_t, as well as $a_{y,1}$, $a_{y,2}$, a_r, a_0, a_g are known (or have been estimated). Then, formulate a local level model involving z_t analogous to (4), but for the misspecified Stage 2 model, yielding:

$$
\begin{aligned}
\overbrace{a_y(L)\tilde{y}_t - a_0 - a_r(L)r_t - a_g g_{t-1}}^{\text{misspecified analogue to } GY_t} &= \overbrace{-a_r(L)z_t}^{\text{analogue to } \beta_t} + \mathring{\nu}_t^{\tilde{y}} && \text{(8a)} \\
\underbrace{-a_r(L)\Delta z_t}_{\substack{\text{analogue} \\ \text{to } \Delta\beta_t}} &= \underbrace{-a_r(L)\varepsilon_t^z}_{\substack{\text{analogue} \\ \text{to } (\lambda/T)\eta_t}}, && \text{(8b)}
\end{aligned}
$$

https://www.newyorkfed.org/medialibrary/media/research/economists/williams/data/HLW_Code.zip.

[4]Notice here also that the trend growth equation in (6d) is misspecified in the Stage 2 model, due to g_{t-2} instead of g_{t-1} being included in the relation, making $\mathring{\varepsilon}_t^{y^*} = \varepsilon_t^{y^*} + g_{t-1} - g_{t-2} = \varepsilon_t^{y^*} + \varepsilon_{t-1}^g$, that is, and MA(1) process. Due to the additional ε_{t-1}^g term in the y_t^* (trend) equation, the covariance matrix of the error terms of the state vector will not be diagonal, that is, uncorrelated, anymore. As the focus here is on the output gap misspecification, I do not discuss this issue any further.

where

$$\mathring{\nu}_t^{\tilde{y}} = \varepsilon_t^{\tilde{y}} - [a_0 + \tfrac{(a_g+4a_r)}{2}(g_{t-1} + g_{t-2}) + \tfrac{a_g}{2}\varepsilon_{t-1}^{g}] \tag{9}$$

from (7c) is the misspecified 'error term' in the local level model in (8).[5] The corresponding signal-to-noise ratio is:

$$\lambda_z = \frac{\lambda}{T} = \frac{\bar{\sigma}(-a_r(L)\Delta z_t)}{\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}})} = \frac{a_r(1)\sigma_z}{\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}})} = \frac{a_r\sigma_z}{\bar{\sigma}(\mathring{\nu}_t^{\tilde{y}})}, \tag{10}$$

which now requires the evaluation of the long-run standard deviation of $\mathring{\nu}_t^{\tilde{y}}$ in the denominator. Note here that, even if $(a_g + 4a_r) = 0$ in the data, the long-run standard deviation of $\mathring{\nu}_t^{\tilde{y}}$ will also depend on $\frac{a_g}{2}\sigma_g$ because of the extra $\frac{a_g}{2}\varepsilon_{t-1}^{g}$ term in $\mathring{\nu}_t^{\tilde{y}}$, yielding:

$$\lambda_z = \frac{\lambda}{T} = \frac{a_r\sigma_z}{(\sigma_{\tilde{y}} + a_g\sigma_g/2)}. \tag{11}$$

Evidently, MUE applied to HLW's 'misspecified' Stage 2 model shown on the right hand side of (6) cannot recover the ratio of interest $\lambda_z = \frac{a_r\sigma_z}{\sigma_{\tilde{y}}}$. Estimating the full Stage 3 model with σ_z replaced by $\frac{\hat{\lambda}_z\sigma_{\tilde{y}}}{a_r}$ in the Kalman Filter recursions that compute the likelihood function as implemented in HLW is therefore unsound.

2.4. Implementation of the structural break regressions

The above discussion has focused on outlining the effect of the 'unnecessary misspecification' of the output gap equation in the Stage 2 model on the relation between λ_z and the recovered signal-to-noise ratio. In this section I describe how Holston *et al.* (2017) 'modify' the structural break dummy variable regressions in their Stage 2 MUE implementation from Stock and Watson's (1998) format. This modification is highly consequential, because it excessively and spuriously amplifies the size of the F statistics obtained from these structural break tests, which are subsequently inverted to produce an estimate of λ_z.

As a reminder, Stock and Watson (1998) first fit an AR(4) model to the GDP growth variable GY_t in (2a).[6] Then, for each $\tau \in [\tau_0, \tau_1]$, they test for a structural break in the unconditional mean of the AR(4) filtered GY_t series (defined as $\hat{a}(L)GY_t$) by running the regression:

$$\hat{a}(L)GY_t = \zeta_0 + \zeta_1 D_t(\tau) + \epsilon_t, \tag{12}$$

where $\hat{a}(L)$ is the estimated counterpart to the AR(4) lag polynomial $a(L)$ in (2c), $D_t(\tau)$ is a dummy variable that is equal to 1 if $t > \tau$, and 0 otherwise, and $\tau = \{\tau_0, \tau_0 + 1, \tau_0 +$

[5]Note here that $g_{t-1} = \frac{1}{2}(g_{t-1} + g_{t-1}) = \frac{1}{2}(g_{t-1} + g_{t-2} + \varepsilon_{t-1}^{g})$ so that $a_g g_{t-1} = \frac{a_g}{2}(g_{t-1} + g_{t-2}) + \frac{a_g}{2}\varepsilon_{t-1}^{g}$.

[6]See Section 3.1 in Buncic (2020) on how the structural break tests in MUE of Stock and Watson (1998) are implemented. Alternatively, see the implementation in Stock and Watson's (1998) GAUSS file `TST_GDP1.GSS` which is available from Mark Watson's homepage at http://www.princeton.edu/~mwatson/ddisk/tvpci.zip, in particular, lines 39 to 66 which AR(4) filter the GDP growth data, and lines 68 to 83 which then implement the Chow (1960) type structural break tests to the AR(4) filtered data.

$2, \ldots, \tau_1\}$ is an index (or sequence) of grid points between endpoints τ_0 and τ_1. Stock and Watson (1998) set these endpoints at the 15^{th} and 85^{th} percentiles of the sample size T, that is, $\tau_0 = 0.15T$ and $\tau_1 = 0.85T$.[7] The sequence of F statistics $(\{F(\tau)\}_{\tau=\tau_0}^{\tau_1})$ on the $\hat{\zeta}_1(\tau)$ point estimate (for each $\tau \in [\tau_0, \tau_1]$) is then utilized in the computation of Andrews and Ploberger's (1994) mean Wald (MW) and exponential Wald (EW) tests, and also Quandt's (1960) Likelihood ratio (QLR) test as:

$$\text{MW} = \frac{1}{N_\tau} \sum_{\tau=\tau_0}^{\tau_1} F(\tau) \tag{13a}$$

$$\text{EW} = \ln\left(\frac{1}{N_\tau} \sum_{\tau=\tau_0}^{\tau_1} \exp\left\{\frac{1}{2}F(\tau)\right\}\right) \tag{13b}$$

$$\text{QLR} = \max_{\tau\in[\tau_0,\tau_1]} \{F(\tau)\}_{\tau=\tau_0}^{\tau_1}, \tag{13c}$$

respectively.[8] Note that $\hat{a}(L)$ is known before implementing the structural break dummy variable regressions in (12). The left hand side variable $\hat{a}(L)GY_t$ is the same for all $\tau \in [\tau_0, \tau_1]$ and $\hat{a}(L)$ does not change or vary with $\tau \in [\tau_0, \tau_1]$.

Stock and Watson (1998) also compute Nyblom's (1989) L structural break test statistic, which is constructed directly from the sum of squared cumulative sums of the demeaned $\hat{a}(L)GY_t$ series and therefore does not require a dummy variable regression in the form of (12) as for the Chow (1960) type break tests in (13).[9] From the Asymptotic Relative Efficiencies (AREs) of the various MUEs reported in Table 2 on page 353 of Stock and Watson (1998) it is apparent that MUE based on Nyblom's (1989) L test statistic has the lowest AREs, suggesting that it is the most efficient estimator. Moreover, Stock and Watson's (1998) estimates of the parameters of the local-level trend growth model reported in Table 5 on page 354 are based on the MUE from Nyblom's (1989) L statistic with $\hat{\sigma}_{\Delta\beta} = 0.13$, suggesting, by revealed preferences, that this is their 'preferred' estimator.[10] In the current setting, I will use Nyblom's (1989) L statistic based MUE of λ_z as a baseline 'order of magnitude' to which the other test statistics can be compared to.[11]

In Holston *et al.*'s (2017) 'modified' implementation of the structural break regressions, extra 'unnecessary' conditioning variables are included on the right hand side of (12). To understand why these are unnecessary, notice that after having estimated the misspecified

[7] More precisely, τ_0 is computed as `floor(0.15 * T)` and τ_1 as $T - \tau_0$ in their GAUSS code. In HLW, these are set at $\tau_0 = 4$ and $\tau_1 = T - 4$, respectively.

[8] Recall that the MW, EW and QLR tests are Chow (1960) type structural break tests, which test the unconditional mean of a series for a structural break at a given or known point in time τ, which defines the point where the data are partitioned into two sub-periods.

[9] See equation 12 in Buncic (2020) for details on how the L test statistic is constructed.

[10] See the Note below Table 4 on page 353 in Stock and Watson (1998) that $\hat{\sigma}_{\Delta\beta}$ is constructed from $T^{-1}\hat{\lambda}\hat{\sigma}_\varepsilon/\hat{a}(1)$.

[11] Looking at the MUEs from the various structural break tests in Table 4 on page 354 in Stock and Watson (1998) highlights that there is considerable variation in the estimates of λ as well as $\sigma_{\Delta\beta}$ from the six different structural break tests that are employed, yielding point estimates of λ and $\sigma_{\Delta\beta}$ between 0 to 4.1, and 0 to 0.13, respectively. However, the λ and $\sigma_{\Delta\beta}$ point estimates are *largest* from Nyblom's (1989) L test statistic.

Stage 2 model in (6), the coefficients $\hat{a}_{y,1}$, $\hat{a}_{y,2}$, $\hat{a}_r$, $\hat{a}_0$, $\hat{a}_g$ as well as the filtered and smoothed estimates of the latent states $\tilde{y}_t$ and g_t are observed or known. In line with Stock and Watson's (1998) implementation in (12), this would allow us to estimate the following structural break regression:

$$\underbrace{\hat{a}_y(L)\hat{\tilde{y}}_{t|T} - \hat{a}_0 - \hat{a}_r(L)r_t - \hat{a}_g\hat{g}_{t-1|T}}_{\text{observed misspecified analogue to } GY_t \text{ in (8a)}} = \zeta_0 + \zeta_1 D_t(\tau) + \epsilon_t \tag{14}$$

for each $\tau \in [\tau_0, \tau_1]$, with $D_t(\tau)$ the dummy variable defined earlier in (12), and $\hat{\tilde{y}}_{t|T}$ and $\hat{g}_{t-1|T}$ the Kalman smoothed estimates of the output gap $\tilde{y}_t$ and trend growth g_t, respectively.[12] The resulting sequence of F statistics on $\hat{\zeta}_1(\tau)$ is then used in the computation of the MW, EW and QLR structural break tests from which the MUE of λ_z is obtained.

But this is not how Holston *et al.* (2017) implement the dummy variable structural break regressions. Instead of (14), Holston *et al.* (2017) estimate:

$$\hat{\tilde{y}}_{t|T} = a_0 + a_1\hat{\tilde{y}}_{t-1|T} + a_2\hat{\tilde{y}}_{t-2|T} + a_r(r_{t-1} + r_{t-2})/2 + a_g\hat{g}_{t-1|T} + \zeta_1 D_t(\tau) + \epsilon_t \tag{15}$$

That is, rather then using the left hand side analogue to GY_t as in (14) which is pre-computed outside the loop, Holston *et al.* (2017) estimate dummy variable regressions with $\hat{\tilde{y}}_{t|T}$ on the left hand side and add $\hat{\tilde{y}}_{t-1|T}$, $\hat{\tilde{y}}_{t-2|T}$, $(r_{t-1} + r_{t-2})/2$ and $\hat{g}_{t-1|T}$ as '*extra regressors*' on the right hand side of (15). This effectively leads to a 're-estimation' of the output gap parameters for each $\tau \in [\tau_0, \tau_1]$, even though these are already known from the full sample estimate of the Stage 2 model.[13]

Why is this so important to point out? For the misspecified Stage 2 model, these two different implementations of the structural break tests yield vastly different estimates of ζ_1, sequences of F statistics, λ_z estimates and '*other factor*' z_t that determines the natural rate. For the correctly specified Stage 2 model, only minor differences result form these two different implementations. The counterparts to the dummy variable regressions in (14) and (15) for the correctly specified Stage 2 model are, respectively:

$$\underbrace{\hat{a}_y(L)\hat{\tilde{y}}_{t|T} - \hat{a}_r(L)[r_t - 4\hat{g}_{t-1|T}]}_{\text{observed analogue to } GY_t \text{ in (4a)}} = \zeta_0 + \zeta_1 D_t(\tau) + \epsilon_t \tag{16}$$

and

$$\hat{\tilde{y}}_{t|T} = a_1\hat{\tilde{y}}_{t-1|T} + a_2\hat{\tilde{y}}_{t-2|T} + a_r(r_{t-1} + r_{t-2} - 4[\hat{g}_{t-1|T} + \hat{g}_{t-2|T}])/2 + \zeta_1 D_t(\tau) + \epsilon_t. \tag{17}$$

[12]This means that $(\hat{a}_y(L)\hat{\tilde{y}}_{t|T} - \hat{a}_0 - \hat{a}_r(L)r_t - \hat{a}_g\hat{g}_{t-1|T})$ on the left-hand side of (14) is pre-computed outside the dummy variable regression loop.

[13]Note here that Stock and Watson (1998) could have also estimated dummy variable regressions of the form $GY_t = \zeta_0 + a_1 GY_{t-1} + a_2 GY_{t-2} + a_3 GY_{t-3} + a_4 GY_{t-4} + \zeta_1 D_t(\tau) + \epsilon_t$, and then constructed the look-up values in Table 3 based on structural break statistics from this regression. However, they did not. Thus, staying within the settings of their original structural break regression framework when utilizing these tabulated values is important.

Note here that by no means am I suggesting that these are simply 'alternative ways' of implementing the structural break tests. The important point to take away from this discussion is that Holston *et al.*'s (2017) 'modified' implementation together with their 'unnecessary misspecification' of the Stage 2 model yields vastly different sequences of F statistics. Section 3 shows that comparison.

In the remainder of the paper, I will refer to HLW's implementation of the structural break tests in (15) and (17) as 'Time varying ϕ' due to the variation of the $\hat{a}_{y,1}$, $\hat{a}_{y,2}$, $\hat{a}_r$, $\hat{a}_0$, $\hat{a}_g$ coefficients for each $\tau \in [\tau_0, \tau_1]$ in the structural break dummy variable regressions, and the 'original' break test implementation of Stock and Watson (1998) shown in (14) and (16) as 'Constant ϕ'.

3. Empirical results

This section provides the full empirical results of HLW's estimates from the correct Stage 2 model specification and MUE implementation. The original estimates of Holston *et al.* (2017) are also reported for ease of comparability. Some of the results are provided as supplementary information or purely for reasons of completeness/reproducibility, and do not merit any lengthy discussion. In all results, I use the same data as described in Holston *et al.* (2017) (see the replication files for details) with the sample ending in 2019:Q4. Readers not interested in the Stage 2 MUE results per se can skip directly to the plots of the natural rate r_t^*, trend growth g_t, *'other factor'* z_t and the output gap $\tilde{y}_t$ in Figures 2, 5, 8 and 11.

In Tables 1, 4, 7 and 10, the Stage 2 model parameter estimates for the US, the Euro Area, the UK and Canada are reported. The first column ('HLW.R-File') provides HLW's estimates obtained from their R-Files. The second column ('HLW($\hat{\sigma}_g^{\mathrm{MLE}}$)') reports estimates from HLW's misspecified Stage 2 model, but with σ_g estimated by MLE rather than MUE in the first stage. The last column ('Correct') lists the estimates of the correctly specified Stage 2 model defined in the left column block of (6), where σ_g is again estimated directly by MLE. Values in round brackets are implied from the $\lambda_g = \sigma_g/\sigma_{y^*}$ relation. Notice here that the MLE of σ_g does not shrink towards zero, and is in fact larger than the MUE implied estimate for three out of the four countries, making estimation of the Stage 1 model redundant.[14]

In Figures 1, 4, 7 and 10 the sequence of F statistics from the dummy variable regressions of HLW's misspecified (top panel) and the 'correctly' specified (bottom panel) Stage 2 models are plotted under the 'Time varying ϕ' and 'Constant ϕ' settings defined in (14) to (17). Examining initially the F statistics from the 'correctly' specified Stage 2 model in the bottom panel, it is clear how similar the 'Time varying ϕ' and 'Constant ϕ' implementations are for all four countries. Both implementations suggest low values of the F statistics, which translate to small values of λ_z and hence σ_z. From HLW's misspecified Stage 2 model shown in

[14] Although the Euro Area and the UK are not 'countries', I will simply use 'the four countries' to mean the estimates for the US, the Euro Area, the UK, and Canada to avoid cumbersome language.

the top panel, however, the 'Time varying ϕ' and 'Constant ϕ' implementations yield vastly different values, most strongly so for the US and Canada, with the 'Time varying ϕ' implementations being excessively larger then those from the 'Constant ϕ' implementation. Note here again that the purpose of the comparison is simply to show that for the misspecified Stage 2 model, the 'Time varying ϕ' implementation excessively amplifies the F statistics. As discussed earlier in Section 2, there is no reason to re-estimate the parameters in the dummy variable regressions, since they are already known from the Stage 2 model estimates. Moreover, it is inconsistent with the way the look-up values in Table 3 of Stock and Watson (1998) were generated.

Tables 2, 5, 8 and 11, show how the different sequences of F statistics translate into different structural break tests and also MUEs of λ_z. The tables are arranged in two column blocks, showing the outputs from the 'Time varying ϕ' and 'Constant ϕ' implementations in the left and right blocks, respectively. In the top part of the tables, the λ_z estimates together with 90% confidence intervals are reported, with the bottom part showing corresponding structural break test statistics together with $p-$values in parenthesis. The individual model headings are the same as in the tables showing the Stage 2 estimates, that is, 'HLW.R-File', 'HLW($\hat{\sigma}_g^{\text{MLE}}$)' and 'Correct'. Note here that the goal of showing the 'HLW($\hat{\sigma}_g^{\text{MLE}}$)' results is to highlight how different the MW, EW and QLR structural break tests and MUEs of λ_z are under the 'Time varying ϕ' implementation provided in the left column block from those obtained from Nyblom's (1989) L test statistic, which does not rely on dummy variable regressions. Nyblom's (1989) L statistics are highly insignificant, with corresponding $p-$values of between 0.69 and 0.945, resulting in MUEs of λ_z that are exactly zero.[15] The MW, EW and QLR tests, on the other hand, suggest sizeable non-zero point estimates, with most of them nonetheless, being insignificant. Under the 'Constant ϕ' implementation, all four tests point to an essentially zero estimate of λ_z.

The parameter estimates of the full model (the Stage 3 model) in (1) are reported in Tables 3, 6, 9 and 12. The first column under the heading ('HLW.R-File') lists again the estimates from HLW's R-Files as reference values. The second column ('MLE($\sigma_g|\hat{\lambda}_z^{\text{Correct}}$)') shows estimates when conditioning on the 'Correct' Stage 2 MUE of $\hat{\lambda}_z$ (based on the EW structural break test) and estimating σ_g together with the other parameters by MLE. In the last column ('MLE(σ_g, σ_z)'), all parameters of the full model (including σ_g and σ_z) are estimated by MLE, that is, without using the Stage 2 model's λ_z estimate. Although I provide these results purely for reasons of completeness and without wanting to discuss them at length, there are two points that are worthy to highlight here. First, as in the Stage 2 model, the MLEs of σ_g in the full (Stage 3) model do not shrink to zero, and are, with the exception of the UK estimates, again larger than their MUE based counterparts. For the US, they are 43% larger, and for the Euro Area, even 83% larger. This confirms once more that the Stage 1 model is

[15] A value of exactly zero for λ_z is obtained whenever the structural break test statistic is smaller than the entries corresponding to $\lambda = 0$ in the first row of Table 3 in Stock and Watson's (1998) look-up values, which is 0.118 for Nyblom's (1989) L test.

unnecessary, at least for this data set and sample period.

Second, and perhaps somewhat unexpectedly, the MLE and MUE based estimates of σ_z are very similar when conditioning on the estimate of λ_z from the correctly specified Stage 2 model and correct MUE implementation. For instance, for the US, the non-zero ML estimate of σ_z and the one implied from the relation $\sigma_z = \hat{\lambda}_z^{\text{Correct}} \sigma_{\tilde{y}} / a_r$ are 0.0656 and 0.0623, respectively. Not only does the ML estimate not shrink to zero, thereby providing an efficient reference value, its point estimate is larger than the one implied by MUE. For the other three countries, both, MLE and MUE based estimates of σ_z yield point estimates of zero. It should be stressed here again that Holston *et al.* (2017) do *not* estimate the initial condition of the state vector. As discussed in more detail in Sections 3 and 4 in Buncic (2020), one would expect the difference between the '*pile-up at zero*' frequencies of MUE and MLE without estimating the initial condition (referred to as Maximum Marginal Likelihood Estimator (MMLE) in Stock and Watson (1998)) to be much smaller than when the initial condition is estimated as well. This is a fundamental result of Stock and Watson's (1998) findings and is reflected in the empirical estimates reported in Table 5 on page 354 of their paper; that is, the MMLE of $\sigma_{\Delta\beta}$ does *not* shrink to zero. The results from the last two columns of the Stage 3 estimates are thus consistent with this view.

As a final comment here, notice that for the US, both, the MLE and MUE point estimates yield a non-zero value of $\hat{\sigma}_z$. However, examining the $p-$values of the L, MW, EW and QLR structural break test statistics as well as the corresponding 90% confidence intervals (CIs) of λ_z suggests that these are not statistically different from zero. In line with these findings, the standard error of the ML estimate is also rather large at 0.1145, leading to the same qualitative conclusion.[16] Therefore, even for the US, it seems highly unlikely that '*other factor*' z_t plays an important role. In Holston *et al.*'s (2017) model, the strong and persistent downward trend in '*other factor*' z_t is due to the misspecification of the Stage 2 model and the way the structural break tests are implemented.

Last, I show plots of the (inefficient) Kalman Filter based estimates of the natural rate r_t^*, annualized trend growth g_t, '*other factor*' z_t, and the output gap $\tilde{y}_t$ in Figures 2, 5, 8 and 11. In Figures 3, 6, 9 and 12, Kalman Smoothed estimates are plotted as efficient analogues to the filtered ones.[17] Examining these sets of estimates, it is evident that the largest impact of the correction on the natural rate of interest is through the filtered and smoothed estimates of '*other factor*' z_t. For the UK and Canada, the MLE($\sigma_g | \hat{\lambda}_z^{\text{Correct}}$) trend growth and output gap estimates are virtually unchanged from the estimates reported in HLW's R-Files, while the estimates of '*other factor*' z_t resemble a flat line that is centered at zero. For the Euro Area, these estimates of '*other factor*' z_t are also flat, but are centered somewhat below zero.

[16]Even though it is not clear if one should conduct standard inference here due to the proximity of the point estimate to the parameter boundary of zero, we can see that two standard deviations from the point estimate would yield a negative 95% lower bound on σ_z. The standard error on σ_g is rather small at 0.0235, suggesting a relatively precise estimate of the standard deviation of the increment to trend growth.

[17]As discussed in Harvey (1989) page 151, the mean squared error (MSE) of the filtered states will in general be larger than the MSE of the smoothed states.

Due to the larger point estimate of σ_g from MLE, the variation, and in particular, the drop in trend growth following the global financial crisis and the European sovereign debt crisis is more pronounced than from HLW's R-Files. This is irrespective of whether conditioning on $\hat{\lambda}_z^{\text{HLW}}$ or $\hat{\lambda}_z^{\text{Correct}}$. The excessive variability in HLW's estimate of *'other factor'* z_t for the Euro Area stands out, particularly when compared to the one from the 'Correct' Stage 2 MUE implementation. For the US, the estimates of trend growth and the output gap follow similar trajectories, despite the ML estimate of σ_g being about 43% larger than the one from HLW's R-Files. For trend growth, there is a small wedge between the estimates from about 2008:Q1 until 2014:Q1, while for the output gap, the series start to diverge slowly from 2009:Q1 until the end of the sample in 2019:Q4. The estimates of *'other factor'* z_t begin to diverge consistently from approximately 2005 onwards, and excessively so in the aftermath of the global financial crisis. All in all, HLW's estimates from their misspecified Stage 2 model exaggerate the downward movement in *'other factor'* z_t by approximately 100 basis point.

4. Conclusion

This paper provides estimates of the natural rate of interest from Holston *et al.*'s (2017) structural model, taking into account the correction to the specification of the Stage 2 model and the implementation of the structural break tests described in detail in Buncic (2020). After implementing this correction, the downward trending behaviour of *'other factor'* z_t vanishes entirely for the Euro Area, the UK, and Canada, and is heavily subdued for the US. For the Euro Area, the UK, and Canada, estimates of *'other factor'* z_t resemble a horizontal line that is centered at zero. The implication of these estimates from Holston *et al.*'s (2017) model is that the natural rate of interest is considerably larger than originally reported.

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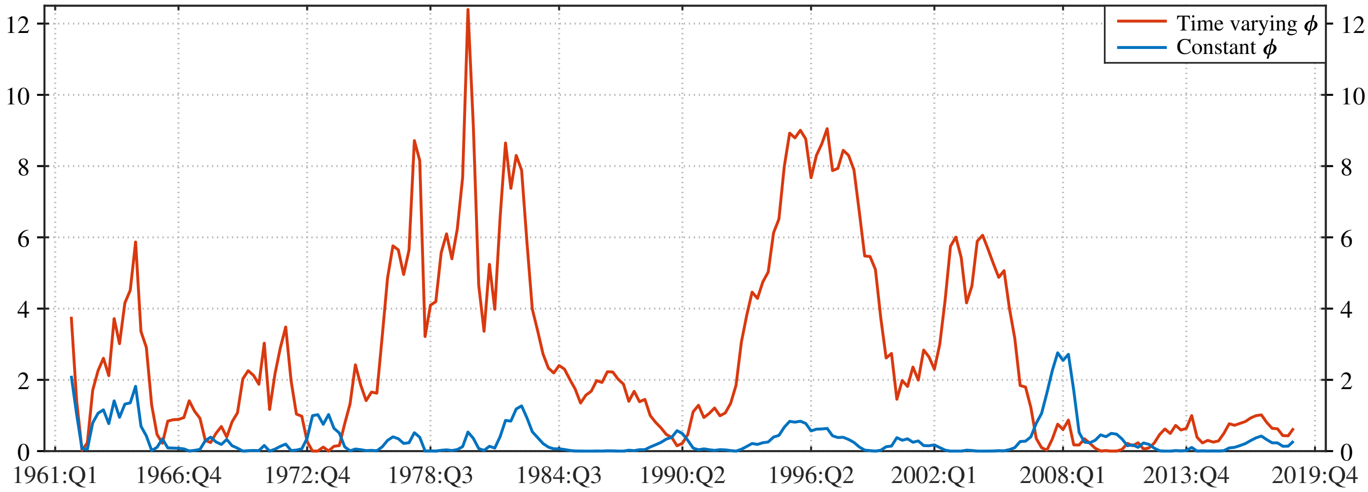
Figures and Tables

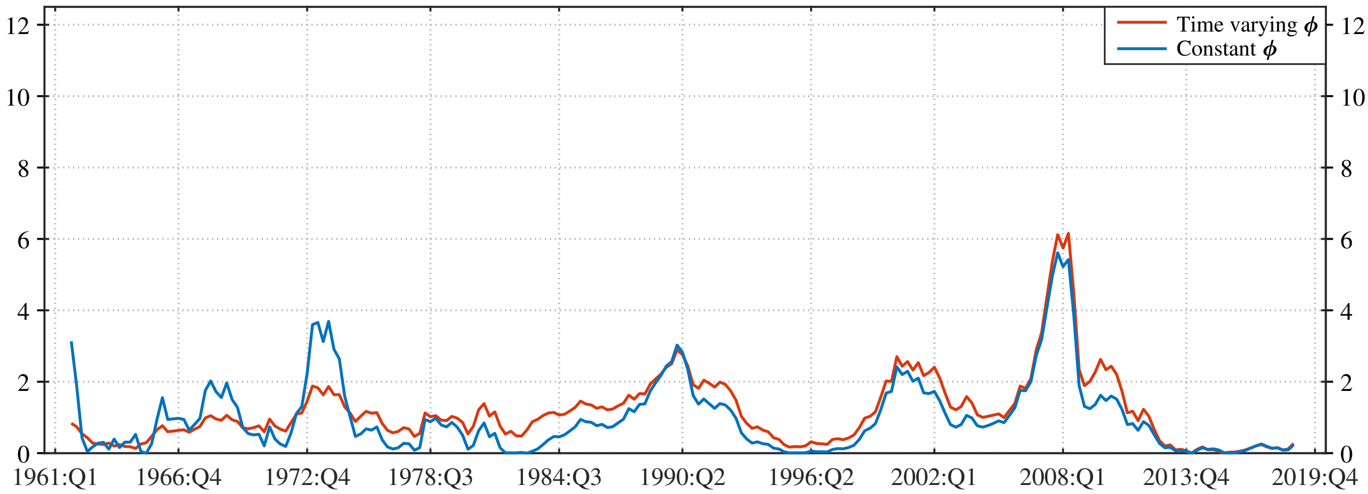
Table 1: Stage 2 parameter estimates for the US

θ_2	HLW.R-File	HLW($\hat{\sigma}_g^{\text{MLE}}$)	Correct
$a_{y,1}$	1.51438668	1.47605055	1.51817781
$a_{y,2}$	−0.57128595	−0.53488827	−0.57824261
a_r	−0.07346174	−0.08402097	−0.06825239
a_0	−0.38877981	−0.39331135	—
a_g	0.75724701	0.79591998	—
b_π	0.66838723	0.67179221	0.67465699
b_y	0.07934934	0.07552146	0.07711016
$\sigma_{\tilde{y}}$	0.33550730	0.32902107	0.34846402
σ_π	0.78523542	0.78643124	0.78785012
σ_{y^*}	0.56797409	0.56163864	0.56051398
σ_g (implied)	(0.03042073)	0.04275448	0.04348430
λ_g (implied)	0.05356007	(0.07612453)	(0.07757934)
Log-likelihood	−534.57461094	−534.37024579	−535.95791031

Notes: This table reports parameter estimates of the Stage 2 model. The first column ('HLW.R-File') lists the estimates obtained from Holston *et al.*'s (2017) R-Files, where λ_g is fixed at the first stage estimate and σ_g is implied from the Stage 1 signal-to-noise ratio $\lambda_g = \sigma_g/\sigma_{y^*}$. The second column ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') shows estimates when σ_g is computed together with the other parameters of Holston *et al.*'s (2017) Stage 2 model by MLE. The last column ('Correct') provides estimates of the *"correctly specified"* Stage 2 model defined in the left column block of (6), where σ_g is again estimated directly by MLE. Values in round brackets give the implied values of σ_g or λ_g from the $\lambda_g = \sigma_g/\sigma_{y^*}$ relation when either λ_g or σ_g is estimated.



(a) Misspecified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)r_t - a_g g_{t-1} - a_0$



(b) Correctly specified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)[r_t - 4g_t]$

Figure 1: Sequence of F statistics from the correctly and incorrectly specified Stage 2 models for the US

Table 2: Stage 2 MUE results of λ_z with corresponding structural break test statistics for the US

λ_z	Time varying $\boldsymbol{\phi}$					Constant $\boldsymbol{\phi}$					
	HLW.R-File	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]	HLW	[90% CI]	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.00]	0.012839	[0, 0.07]	0.000000	[0, 0.01]	0.000000	[0, 0.00]	0.012839	[0, 0.07]
MW	0.031855	0.039365	[0, 0.16]	0.015509	[0, 0.07]	0.000000	[0, 0.02]	0.000000	[0, 0.01]	0.011947	[0, 0.07]
EW	0.035415	0.040444	[0, 0.13]	0.015663	[0, 0.07]	0.000000	[0, 0.03]	0.000000	[0, 0.01]	0.013230	[0, 0.07]
QLR	0.044251	0.047740	[0, 0.16]	0.023180	[0, 0.09]	0.000000	[0, 0.04]	0.000000	[0, 0.03]	0.020805	[0, 0.08]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.037097	(0.9450)	0.170077	(0.3350)	0.049609	(0.8750)	0.037097	(0.9450)	0.170077	(0.3350)
MW	2.747739	3.850794	(0.0150)	1.159561	(0.2850)	0.326527	(0.8150)	0.251819	(0.8900)	0.977257	(0.3550)
EW	2.553645	3.184074	(0.0100)	0.775922	(0.2800)	0.199023	(0.7900)	0.148746	(0.8750)	0.681179	(0.3250)
QLR	12.398150	13.725281	(0.0050)	6.156295	(0.1450)	2.759498	(0.5900)	2.185052	(0.7200)	5.613303	(0.1850)

Notes: This table reports the Stage 2 MUE of λ_z obtained from the *"misspecified"* and *"correctly specified"* Stage 2 models. The table is split into left and right column blocks corresponding to the two different structural break test implementations, which are denoted by 'Time varying $\boldsymbol{\phi}$' and 'Constant $\boldsymbol{\phi}$', respectively. The bottom half of the table lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for the *"misspecified"* Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates when σ_g is computed by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the *"correctly specified"* Stage 2 model where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR structural break tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to the structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

Table 3: Stage 3 parameter estimates for the US

θ_3	HLW.R-File	MLE($\sigma_g \mid \hat{\lambda}_z^{\text{Correct}}$)	MLE(σ_g, σ_z)
$a_{y,1}$	1.53991114	1.51796245	1.51804519
$a_{y,2}$	−0.59855575	−0.57839055	−0.57846940
a_r	−0.06786964	−0.06955268	−0.06941225
b_π	0.67083803	0.67182759	0.67187993
b_y	0.07859265	0.07984256	0.07985050
$\sigma_{\tilde{y}}$	0.33378693	0.34501846	0.34535765
σ_π	0.78620285	0.78669831	0.78672365
σ_{y^*}	0.57390968	0.56180867	0.56167781
σ_g (implied)	(0.03073864)	0.04395352	0.04391602
σ_z (implied)	(0.17417262)	(0.06562828)	0.06237003
λ_g (implied)	0.05356007	(0.07823575)	(0.07818721)
λ_z (implied)	0.03541491	0.01323008	(0.01253553)
Log-likelihood	−536.48377160	−535.97760447	−535.97718006

Notes: This table reports the Stage 3 estimates. The first column ('HLW.R-File') gives the estimates from Holston *et al.*'s (2017) R-Files. The second column ('MLE($\sigma_g|\hat{\lambda}_z^{\text{Correct}}$)') shows estimates from the *"correctly specified"* Stage 2 model's MUE of λ_z (based on the EW structural break test), where σ_g is estimated by MLE. The last column ('MLE(σ_g, σ_g)') reports estimates where all parameters, including (σ_g, σ_g), are computed by MLE. Values in round brackets give the implied (σ_g, σ_z) or (λ_g, λ_z) values constructed from the signal-to-noise ratios $\lambda_g = \sigma_g/\sigma_{y^*}$ and $\lambda_z = a_r\sigma_z/\sigma_{\tilde{y}}$.

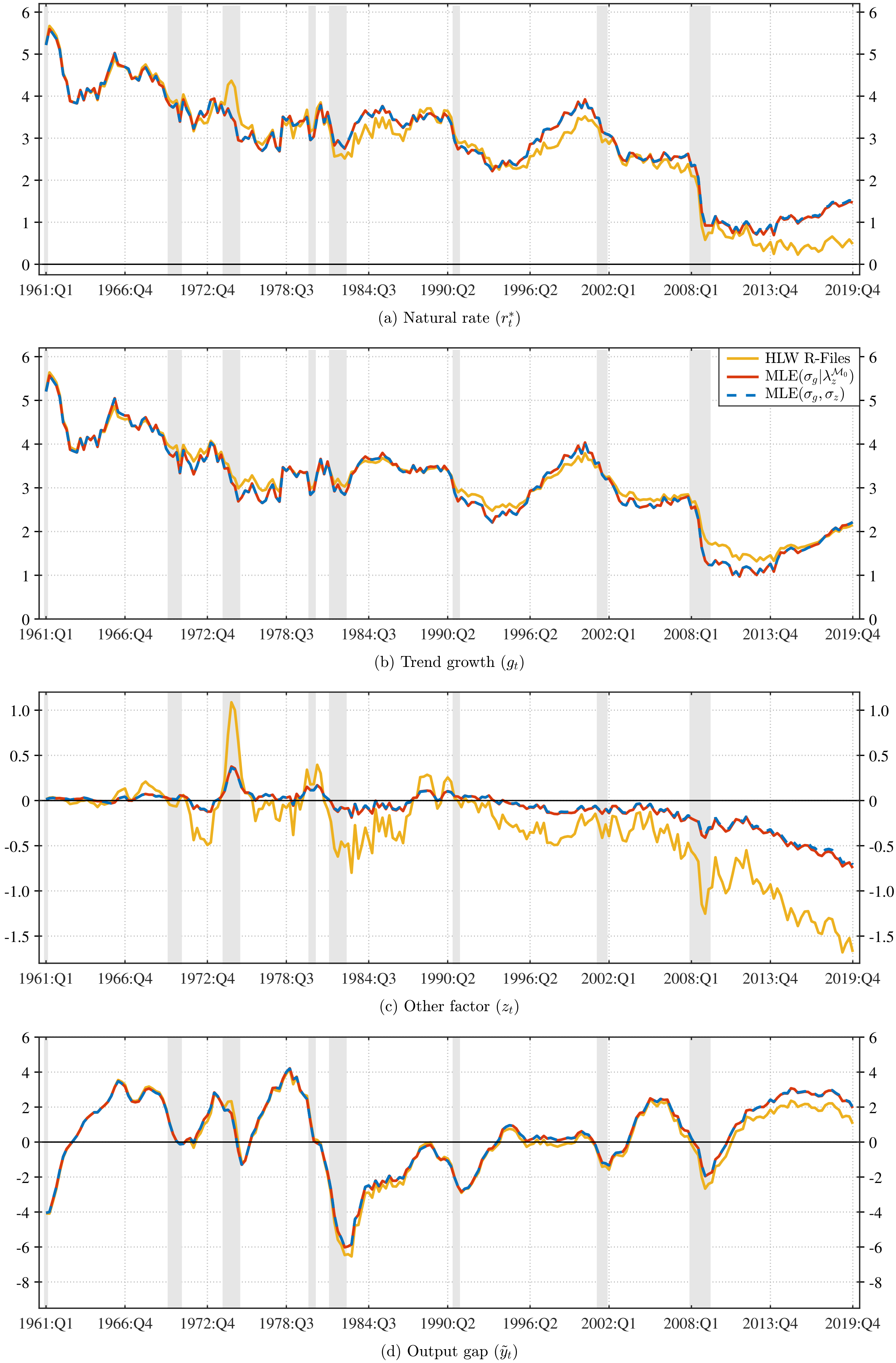
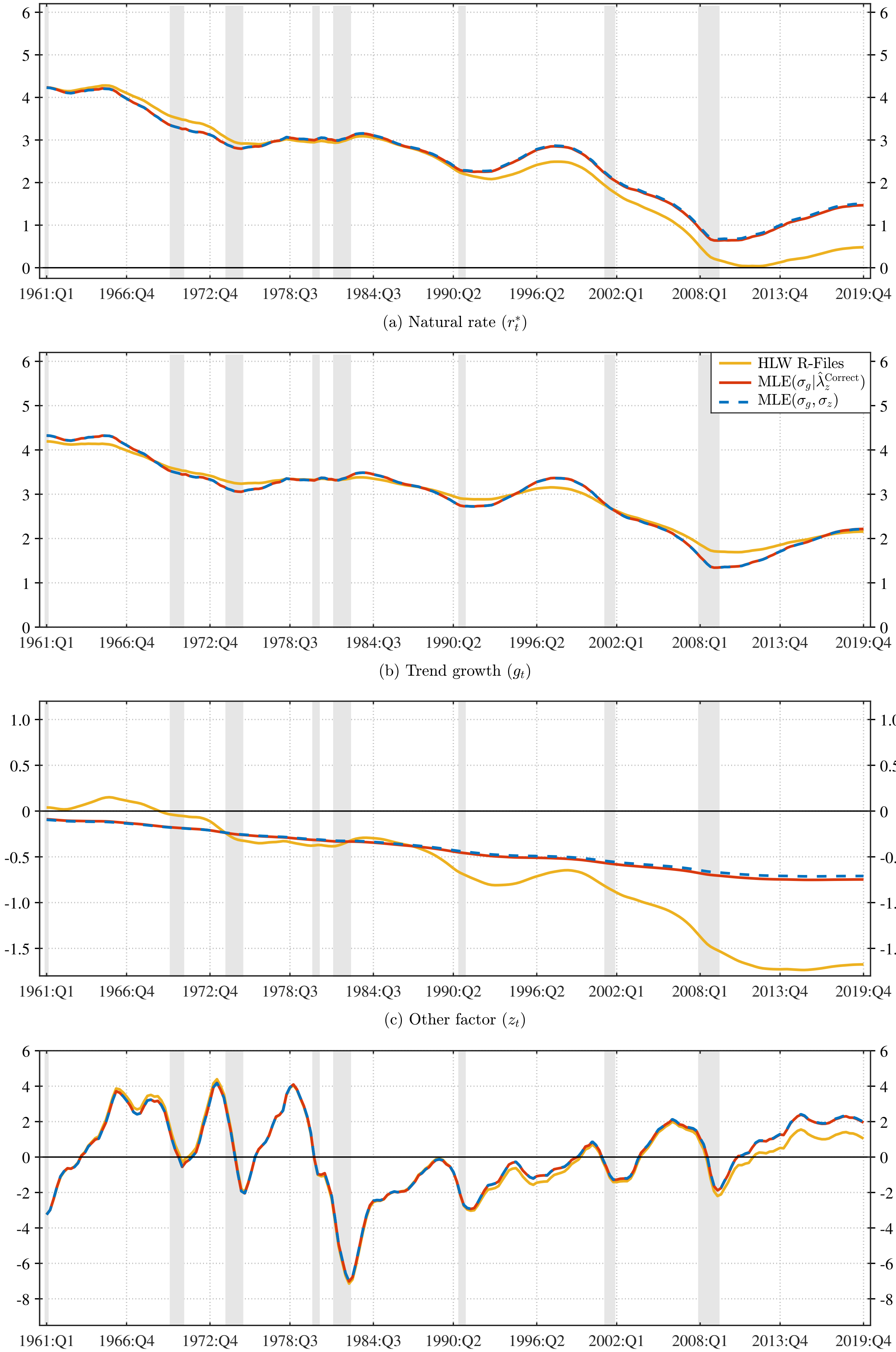


Figure 2: Filtered estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for the US



(a) Natural rate (r_t^*)

(b) Trend growth (g_t)

(c) Other factor (z_t)

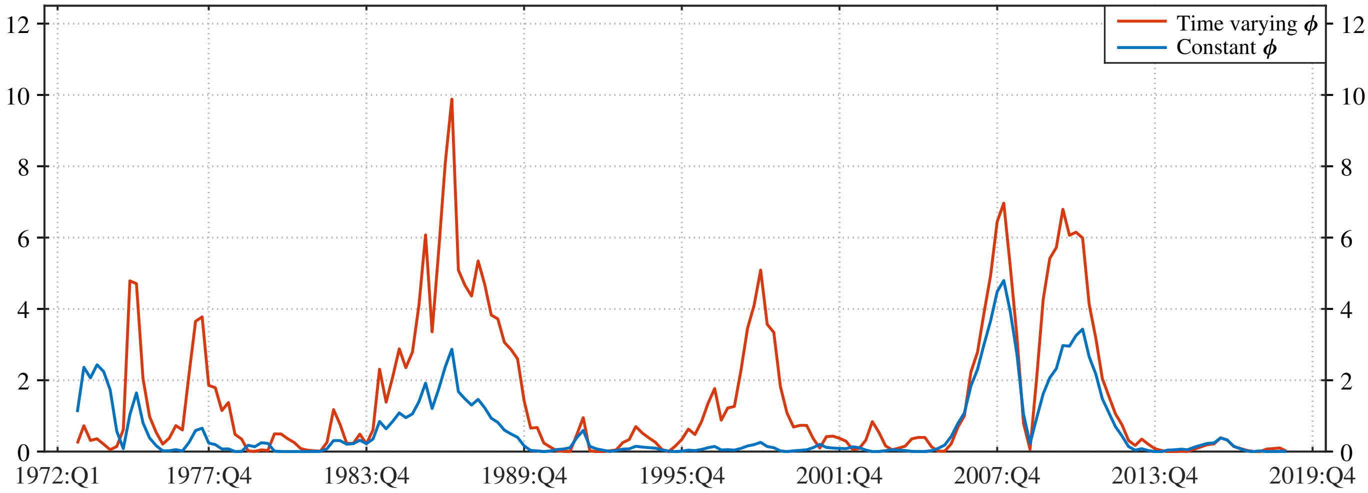
(d) Output gap ($\tilde{y}_t$)

Figure 3: Smoothed estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for the US

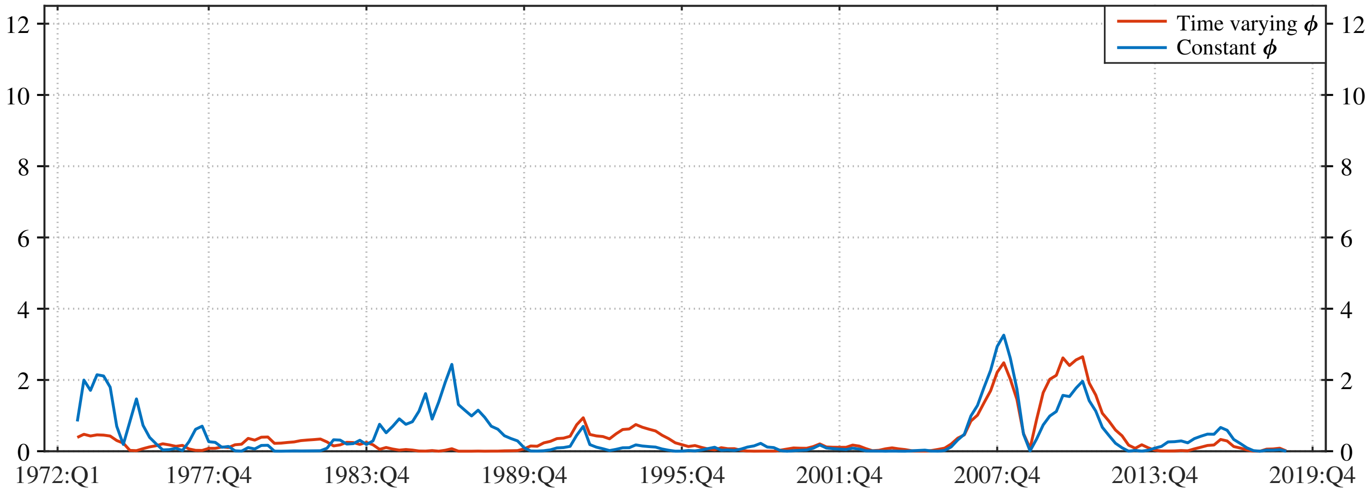
Table 4: Stage 2 parameter estimates for the Euro Area

θ_2	HLW.R-File	MLE(σ_g)	Correct
$a_{y,1}$	1.67569238	1.65132230	1.64793371
$a_{y,2}$	−0.72521483	−0.70073479	−0.69810730
a_r	−0.03196558	−0.03844291	−0.03809637
a_0	−0.03713846	0.00001276	—
a_g	0.18359716	0.12015682	—
b_π	0.67855387	0.68419020	0.68308905
b_y	0.08063733	0.06765010	0.07237654
$\sigma_{\tilde{y}}$	0.29347242	0.31175812	0.31237408
σ_π	0.97845481	0.98011308	0.97997049
σ_{y^*}	0.39318254	0.37062132	0.37015852
σ_g (implied)	(0.01391225)	0.02588260	0.02561294
λ_g (implied)	0.03538370	(0.06983570)	(0.06919452)
Log-likelihood	−422.30013304	−421.13354157	−421.17804744

Notes: This table reports parameter estimates of the Stage 2 model. The first column ('HLW.R-File') lists the estimates obtained from Holston *et al.*'s (2017) R-Files, where λ_g is fixed at the first stage estimate and σ_g is implied from the Stage 1 signal-to-noise ratio $\lambda_g = \sigma_g/\sigma_{y^*}$. The second column ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') shows estimates when σ_g is computed together with the other parameters of Holston *et al.*'s (2017) Stage 2 model by MLE. The last column ('Correct') provides estimates of the *"correctly specified"* Stage 2 model defined in the left column block of (6), where σ_g is again estimated directly by MLE. Values in round brackets give the implied values of σ_g or λ_g from the $\lambda_g = \sigma_g/\sigma_{y^*}$ relation when either λ_g or σ_g is estimated.



(a) Misspecified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)r_t - a_g g_{t-1} - a_0$



(b) Correctly specified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)[r_t - 4g_t]$

Figure 4: Sequence of F statistics from the correctly and incorrectly specified Stage 2 models for the Euro Area

Table 5: Stage 2 MUE results of λ_z with corresponding structural break test statistics for the Euro Area

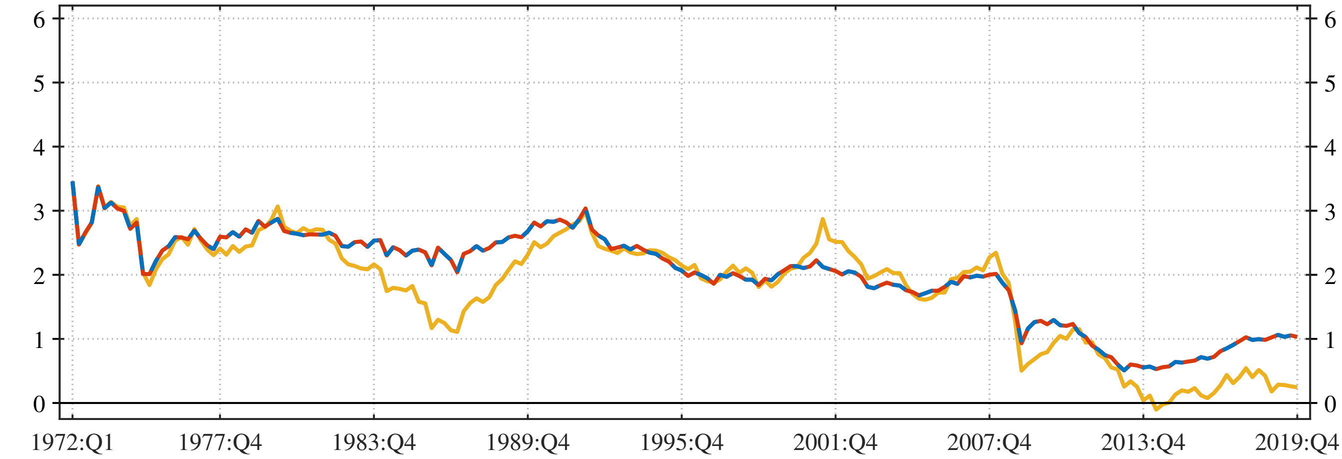
λ_z	Time varying $\boldsymbol{\phi}$					Constant $\boldsymbol{\phi}$					
	HLW.R-File	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]	HLW	[90% CI]	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.05]	0.000000	[0, 0.03]	0.000000	[0, 0.03]
MW	0.024235	0.021309	[0, 0.10]	0.000000	[0, 0.03]	0.000000	[0, 0.06]	0.000000	[0, 0.04]	0.000000	[0, 0.04]
EW	0.032869	0.025664	[0, 0.11]	0.000000	[0, 0.04]	0.005007	[0, 0.07]	0.000000	[0, 0.05]	0.000000	[0, 0.05]
QLR	0.044176	0.040711	[0, 0.14]	0.000000	[0, 0.05]	0.020474	[0, 0.09]	0.006325	[0, 0.07]	0.001430	[0, 0.06]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.067682	(0.7650)	0.068424	(0.7600)	0.097108	(0.5950)	0.067682	(0.7650)	0.068424	(0.7600)
MW	1.493978	1.270379	(0.2500)	0.351069	(0.7900)	0.622436	(0.5450)	0.454903	(0.6850)	0.455205	(0.6850)
EW	1.525650	1.090317	(0.1750)	0.226251	(0.7500)	0.474062	(0.4550)	0.300121	(0.6400)	0.292027	(0.6500)
QLR	9.882686	8.878608	(0.0450)	2.652280	(0.6150)	4.796794	(0.2650)	3.454148	(0.4500)	3.257853	(0.4900)

Notes: This table reports the Stage 2 MUE of λ_z obtained from the *"misspecified"* and *"correctly specified"* Stage 2 models. The table is split into left and right column blocks corresponding to the two different structural break test implementations, which are denoted by 'Time varying $\boldsymbol{\phi}$' and 'Constant $\boldsymbol{\phi}$', respectively. The bottom half of the table lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for the *"misspecified"* Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates when σ_g is computed by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the *"correctly specified"* Stage 2 model where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR structural break tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to the structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

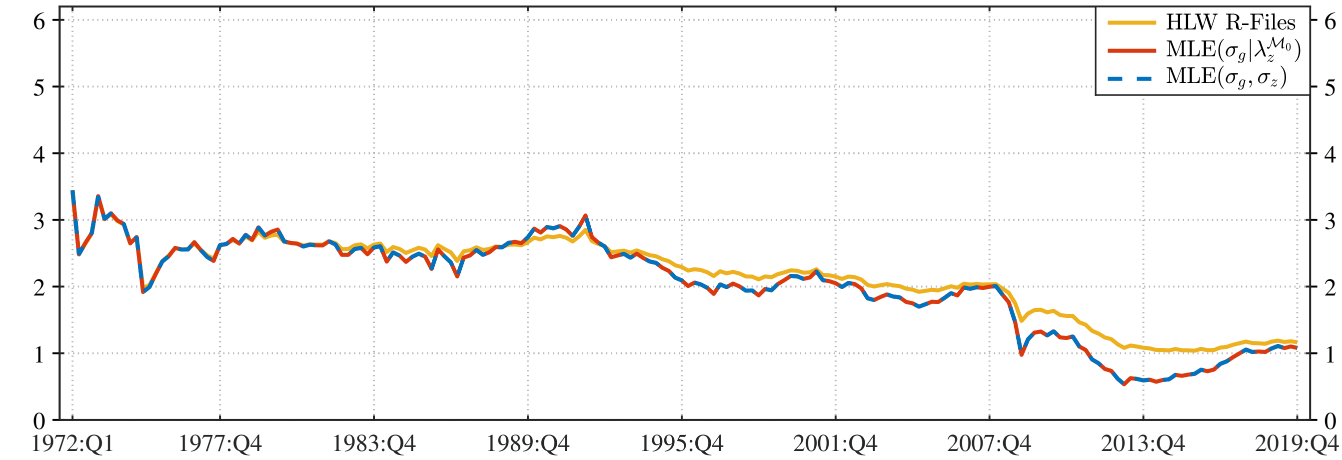
Table 6: Stage 3 parameter estimates for the Euro Area

θ_3	HLW.R-File	MLE($\sigma_g \mid \hat{\lambda}_z^{\text{Correct}}$)	MLE(σ_g, σ_z)
$a_{y,1}$	1.67182426	1.64822320	1.64822319
$a_{y,2}$	−0.72246599	−0.69843910	−0.69843909
a_r	−0.03630451	−0.03767476	−0.03767477
b_π	0.68844980	0.68370936	0.68370935
b_y	0.06512193	0.07134919	0.07134919
$\sigma_{\tilde{y}}$	0.28941500	0.31252152	0.31252152
σ_π	0.98154863	0.98015826	0.98015826
σ_{y^*}	0.39412651	0.37012107	0.37012107
σ_g (implied)	(0.01394565)	0.02559315	0.02559315
σ_z (implied)	(0.26203108)	(0.00000000)	0.00000002
λ_g (implied)	0.03538370	(0.06914804)	(0.06914804)
λ_z (implied)	0.03286945	0.00000000	(0.00000000)
Log-likelihood	−422.87276090	−421.23654260	−421.23654260

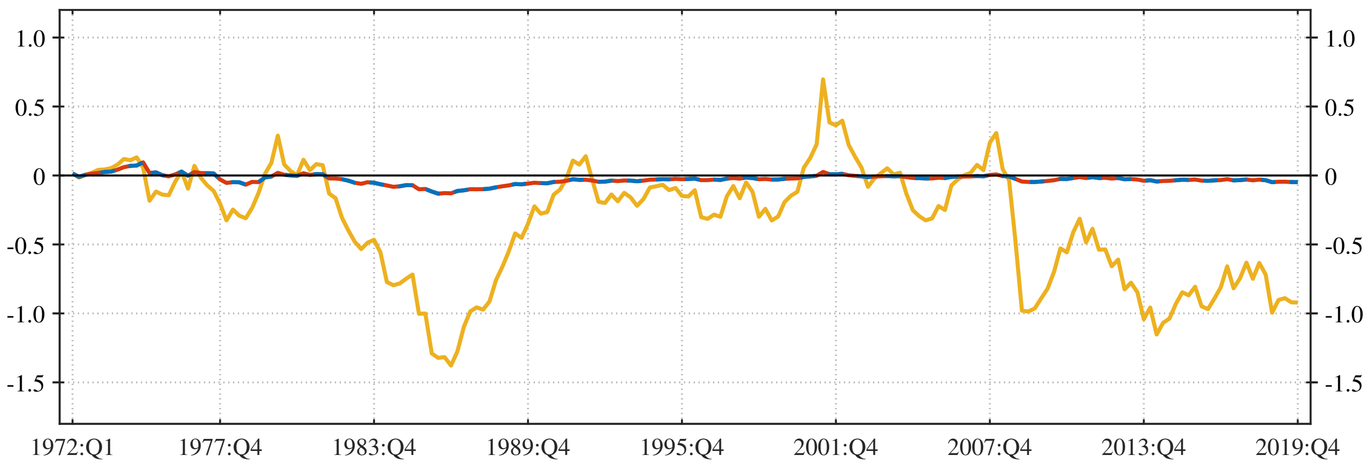
Notes: This table reports the Stage 3 estimates. The first column ('HLW.R-File') gives the estimates from Holston *et al.*'s (2017) R-Files. The second column ('MLE($\sigma_g|\hat{\lambda}_z^{\text{Correct}}$)') shows estimates from the *"correctly specified"* Stage 2 model's MUE of λ_z (based on the EW structural break test), where σ_g is estimated by MLE. The last column ('MLE(σ_g, σ_g)') reports estimates where all parameters, including (σ_g, σ_g), are computed by MLE. Values in round brackets give the implied (σ_g, σ_z) or (λ_g, λ_z) values constructed from the signal-to-noise ratios $\lambda_g = \sigma_g/\sigma_{y^*}$ and $\lambda_z = a_r\sigma_z/\sigma_{\tilde{y}}$.



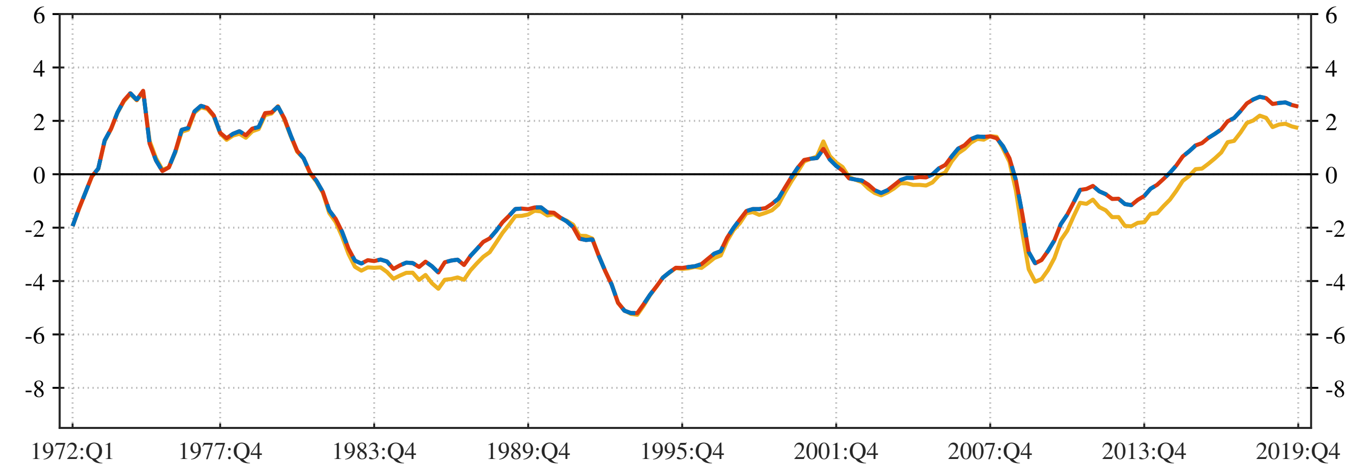
(a) Natural rate (r_t^*)



(b) Trend growth (g_t)

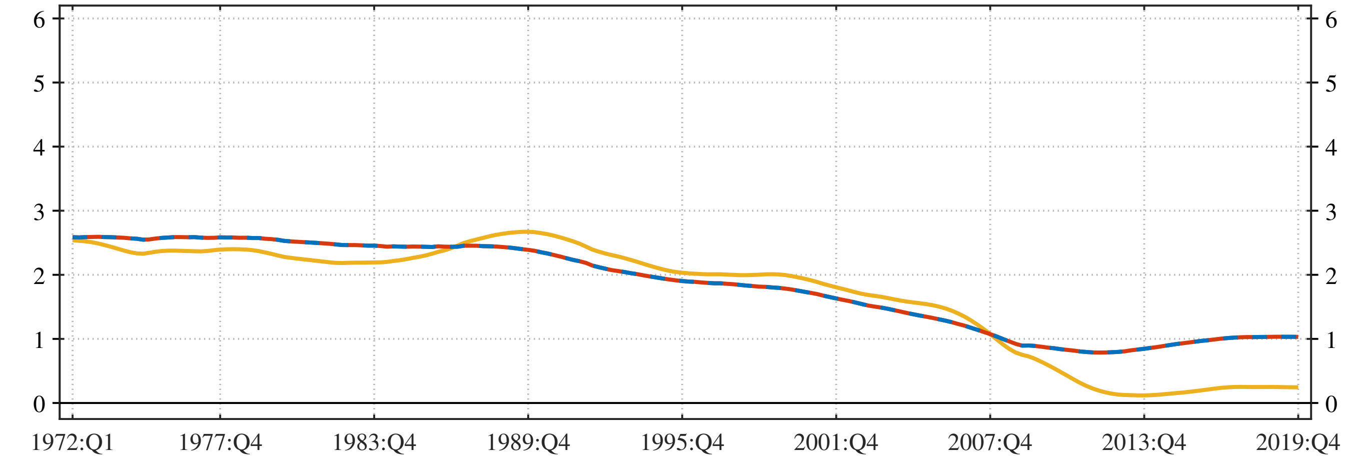


(c) Other factor (z_t)

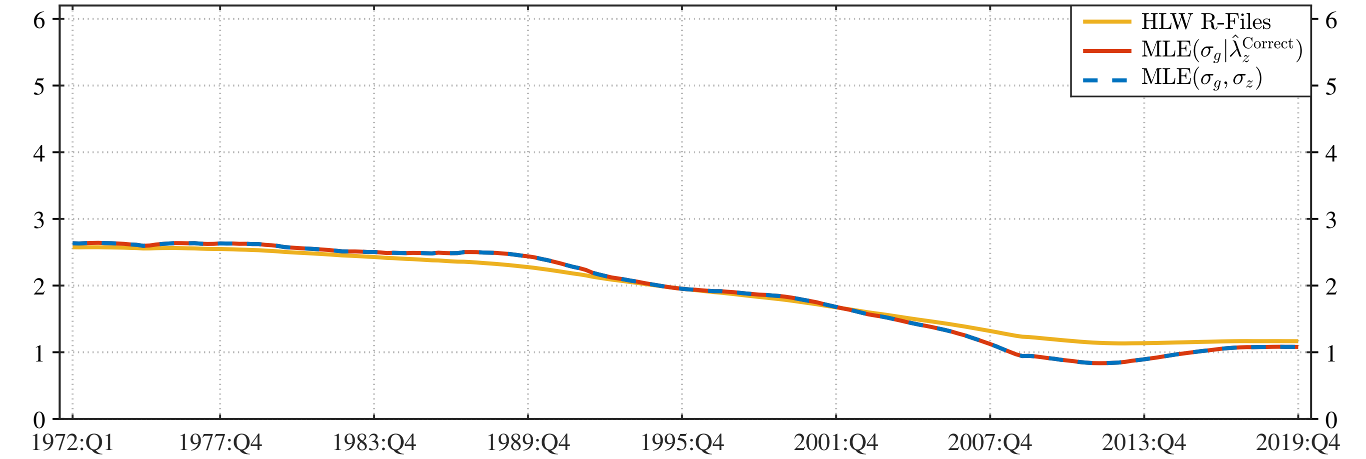


(d) Output gap ($\tilde{y}_t$)

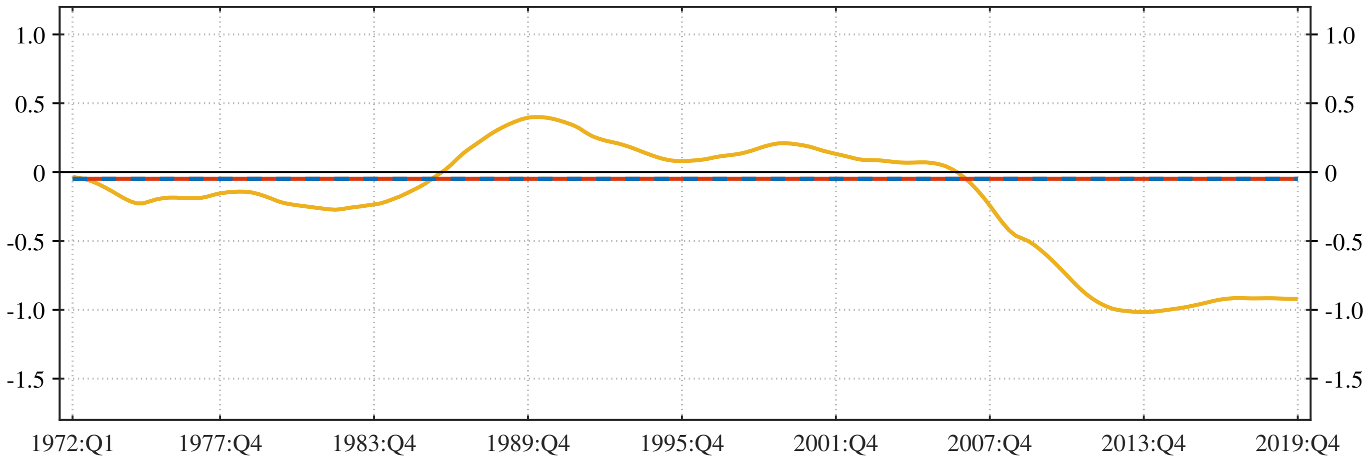
Figure 5: Filtered estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for the Euro Area



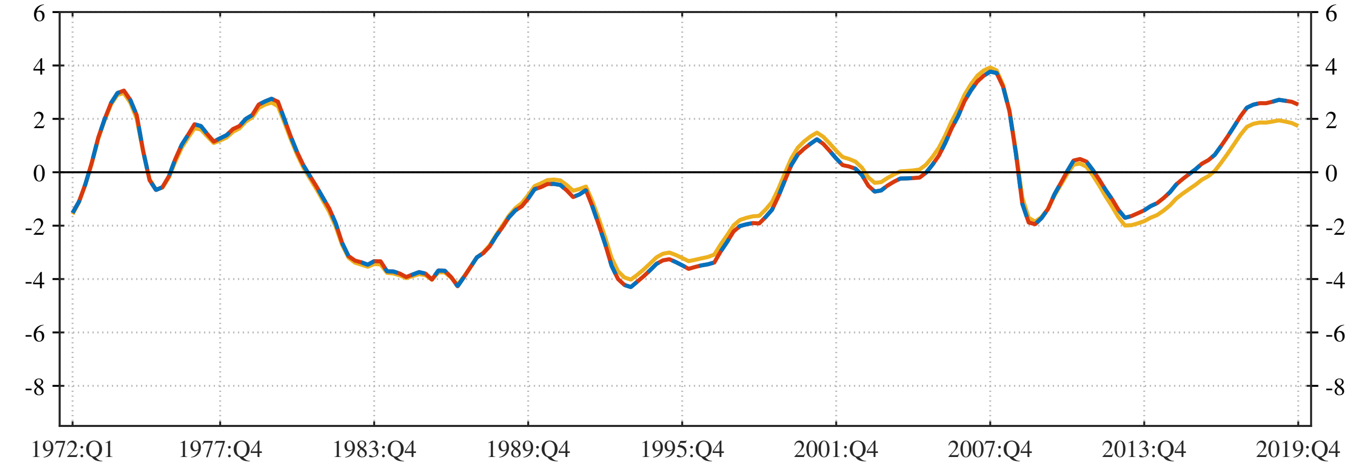
(a) Natural rate (r_t^*)



(b) Trend growth (g_t)



(c) Other factor (z_t)



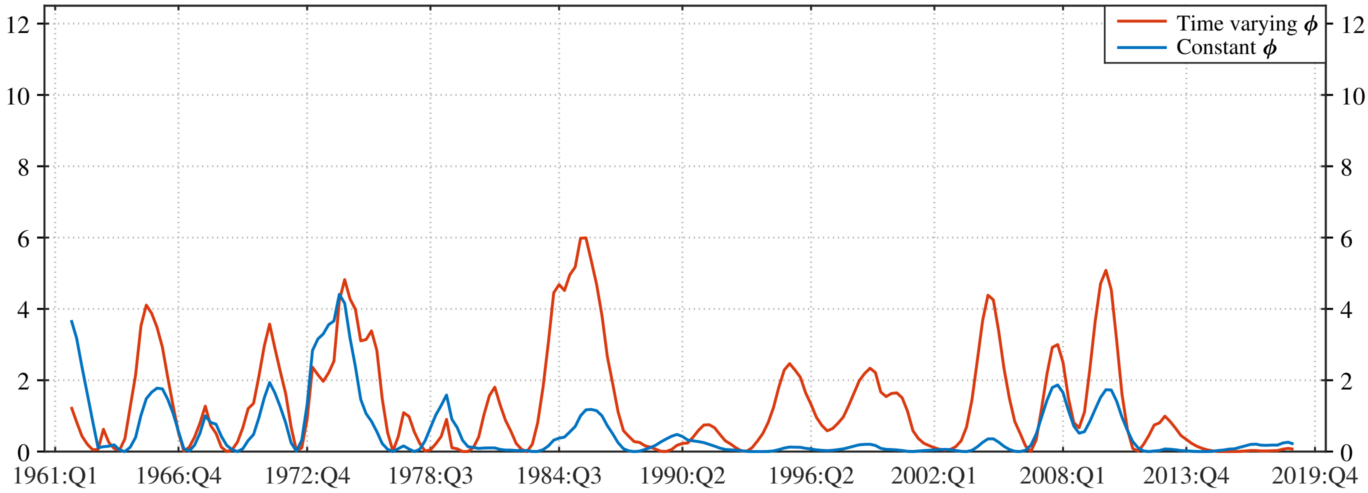
(d) Output gap ($\tilde{y}_t$)

Figure 6: Smoothed estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for the Euro Area

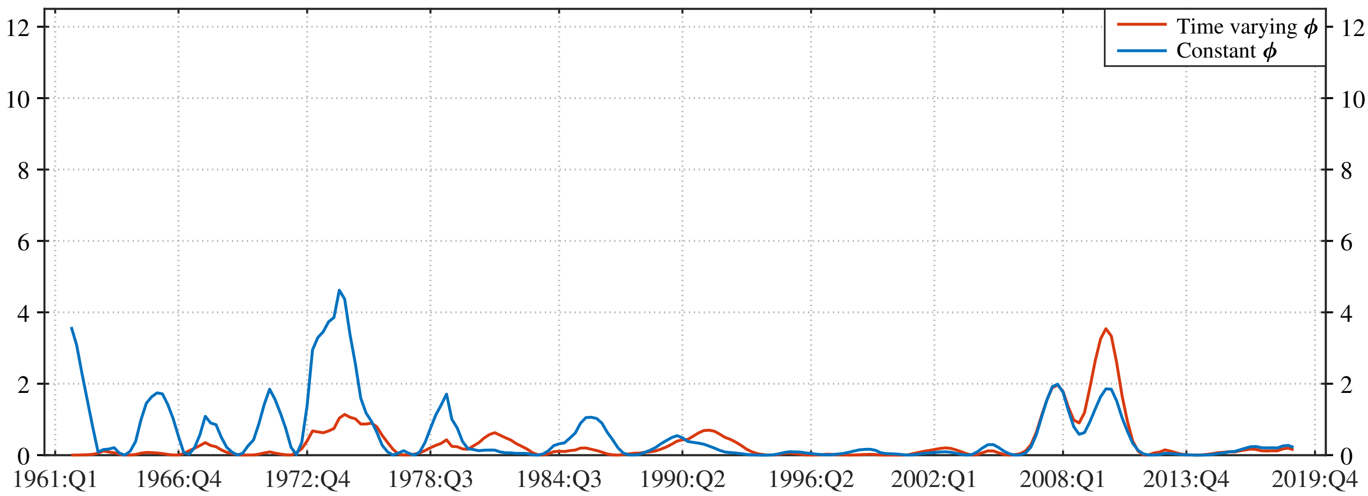
Table 7: Stage 2 parameter estimates for the UK

θ_2	HLW.R-File	MLE(σ_g)	Correct
$a_{y,1}$	1.85797662	1.85829687	1.85897377
$a_{y,2}$	−0.94297465	−0.94345351	−0.94429664
a_r	−0.00844679	−0.00834797	−0.00827683
a_0	−0.01702065	−0.01738428	—
a_g	0.05467354	0.05512076	—
b_π	0.59294069	0.59286362	0.59284733
b_y	0.54294214	0.54451649	0.54717295
$\sigma_{\tilde{y}}$	0.10528595	0.10478315	0.10380441
σ_π	2.64036422	2.64027260	2.64024949
σ_{y^*}	0.85639356	0.85750590	0.85786241
σ_g (implied)	(0.02033304)	0.01808684	0.01819782
λ_g (implied)	0.02374264	(0.02109238)	(0.02121298)
Log-likelihood	−877.75517435	−877.74815473	−877.76354886

Notes: This table reports parameter estimates of the Stage 2 model. The first column ('HLW.R-File') lists the estimates obtained from Holston *et al.*'s (2017) R-Files, where λ_g is fixed at the first stage estimate and σ_g is implied from the Stage 1 signal-to-noise ratio $\lambda_g = \sigma_g/\sigma_{y^*}$. The second column ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') shows estimates when σ_g is computed together with the other parameters of Holston *et al.*'s (2017) Stage 2 model by MLE. The last column ('Correct') provides estimates of the *"correctly specified"* Stage 2 model defined in the left column block of (6), where σ_g is again estimated directly by MLE. Values in round brackets give the implied values of σ_g or λ_g from the $\lambda_g = \sigma_g/\sigma_{y^*}$ relation when either λ_g or σ_g is estimated.



(a) Misspecified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)r_t - a_g g_{t-1} - a_0$



(b) Correctly specified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)[r_t - 4g_t]$

Figure 7: Sequence of F statistics from the correctly and incorrectly specified Stage 2 models for the UK

Table 8: Stage 2 MUE results of λ_z with corresponding structural break test statistics for the UK

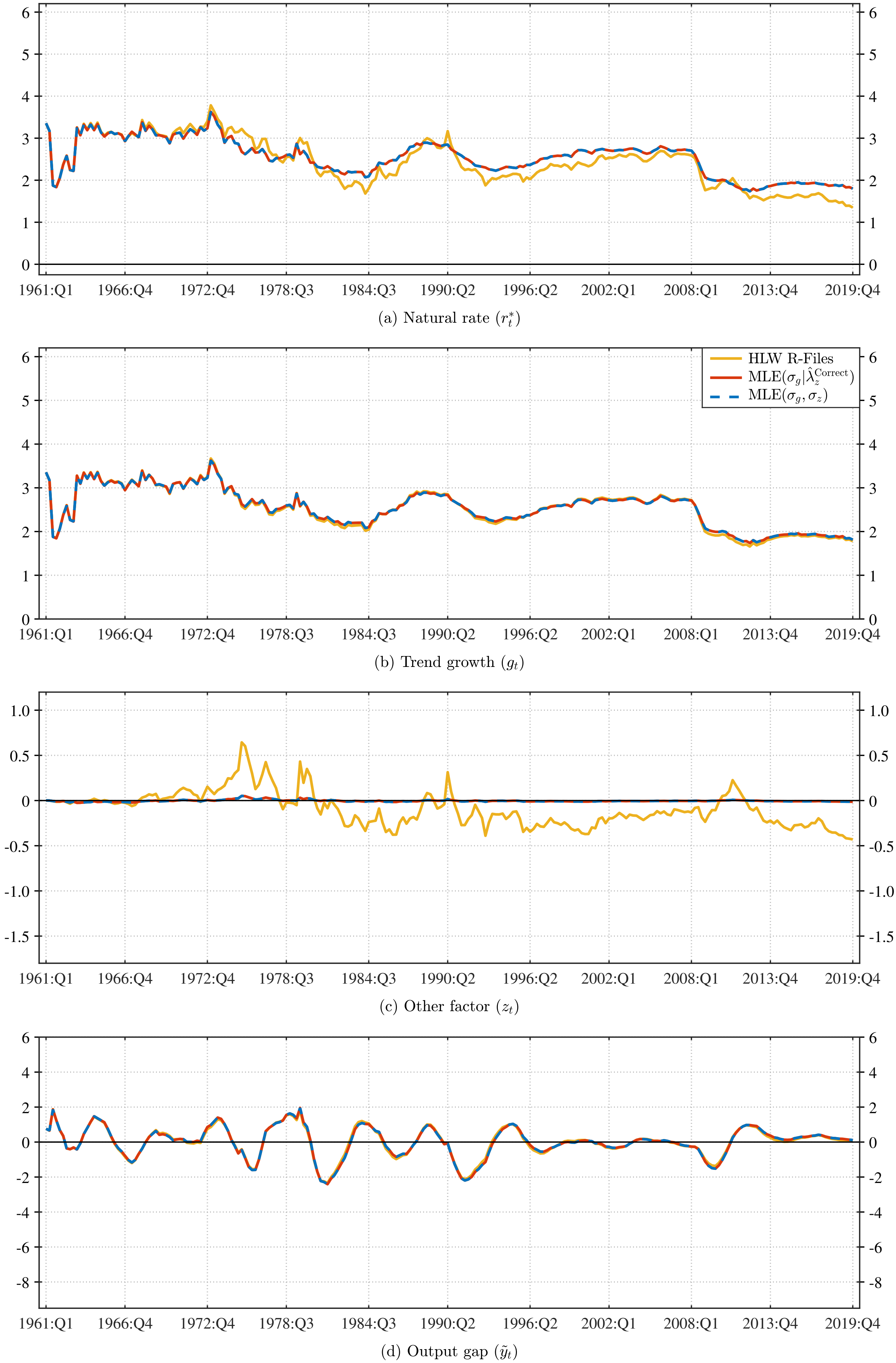
λ_z	Time varying $\boldsymbol{\phi}$					Constant $\boldsymbol{\phi}$					
	HLW.R-File	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]	HLW	[90% CI]	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.03]	0.000000	[0, 0.03]
MW	0.017600	0.017855	[0, 0.08]	0.000000	[0, 0.02]	0.000000	[0, 0.04]	0.000000	[0, 0.04]	0.000000	[0, 0.04]
EW	0.019307	0.019860	[0, 0.08]	0.000000	[0, 0.03]	0.000000	[0, 0.05]	0.000000	[0, 0.05]	0.000000	[0, 0.05]
QLR	0.022484	0.023628	[0, 0.09]	0.007219	[0, 0.05]	0.014437	[0, 0.07]	0.014828	[0, 0.07]	0.015647	[0, 0.07]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.080184	(0.6900)	0.081772	(0.6800)	0.079305	(0.6900)	0.080184	(0.6900)	0.081772	(0.6800)
MW	1.295177	1.319085	(0.2400)	0.309463	(0.8300)	0.540538	(0.6100)	0.546108	(0.6050)	0.555765	(0.6000)
EW	0.984618	1.021808	(0.1900)	0.210030	(0.7750)	0.392172	(0.5350)	0.397702	(0.5300)	0.409986	(0.5150)
QLR	5.993069	6.261275	(0.1400)	3.541268	(0.4350)	4.408112	(0.3050)	4.476558	(0.2950)	4.619925	(0.2800)

Notes: This table reports the Stage 2 MUE of λ_z obtained from the *"misspecified"* and *"correctly specified"* Stage 2 models. The table is split into left and right column blocks corresponding to the two different structural break test implementations, which are denoted by 'Time varying $\boldsymbol{\phi}$' and 'Constant $\boldsymbol{\phi}$', respectively. The bottom half of the table lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for the *"misspecified"* Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates when σ_g is computed by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the *"correctly specified"* Stage 2 model where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR structural break tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to the structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

Table 9: Stage 3 parameter estimates for the UK

θ_3	HLW.R-File	MLE($\sigma_g \mid \hat{\lambda}_z^{\text{Correct}}$)	MLE(σ_g, σ_z)
$a_{y,1}$	1.86610365	1.85954360	1.85954344
$a_{y,2}$	−0.95436885	−0.94521866	−0.94521841
a_r	−0.00710513	−0.00806413	−0.00806420
b_π	0.59266882	0.59271994	0.59271997
b_y	0.56863226	0.55055520	0.55055431
$\sigma_{\tilde{y}}$	0.09058094	0.10269464	0.10269495
σ_π	2.64007048	2.64009312	2.64009319
σ_{y^*}	0.86136819	0.85828956	0.85828943
σ_g (implied)	(0.02045115)	0.01817074	0.01817076
σ_z (implied)	(0.24614394)	(0.00000000)	0.00000001
λ_g (implied)	0.02374264	(0.02117087)	(0.02117089)
λ_z (implied)	0.01930743	0.00000000	(0.00000000)
Log-likelihood	−877.97896968	−877.77522874	−877.77522874

Notes: This table reports the Stage 3 estimates. The first column ('HLW.R-File') gives the estimates from Holston *et al.*'s (2017) R-Files. The second column ('MLE($\sigma_g \mid \hat{\lambda}_z^{\text{Correct}}$)') shows estimates from the *"correctly specified"* Stage 2 model's MUE of λ_z (based on the EW structural break test), where σ_g is estimated by MLE. The last column ('MLE(σ_g, σ_g)') reports estimates where all parameters, including (σ_g, σ_g), are computed by MLE. Values in round brackets give the implied (σ_g, σ_z) or (λ_g, λ_z) values constructed from the signal-to-noise ratios $\lambda_g = \sigma_g/\sigma_{y^*}$ and $\lambda_z = a_r\sigma_z/\sigma_{\tilde{y}}$.



(a) Natural rate (r_t^*)

(b) Trend growth (g_t)

(c) Other factor (z_t)

(d) Output gap ($\tilde{y}_t$)

Figure 8: Filtered estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for the UK

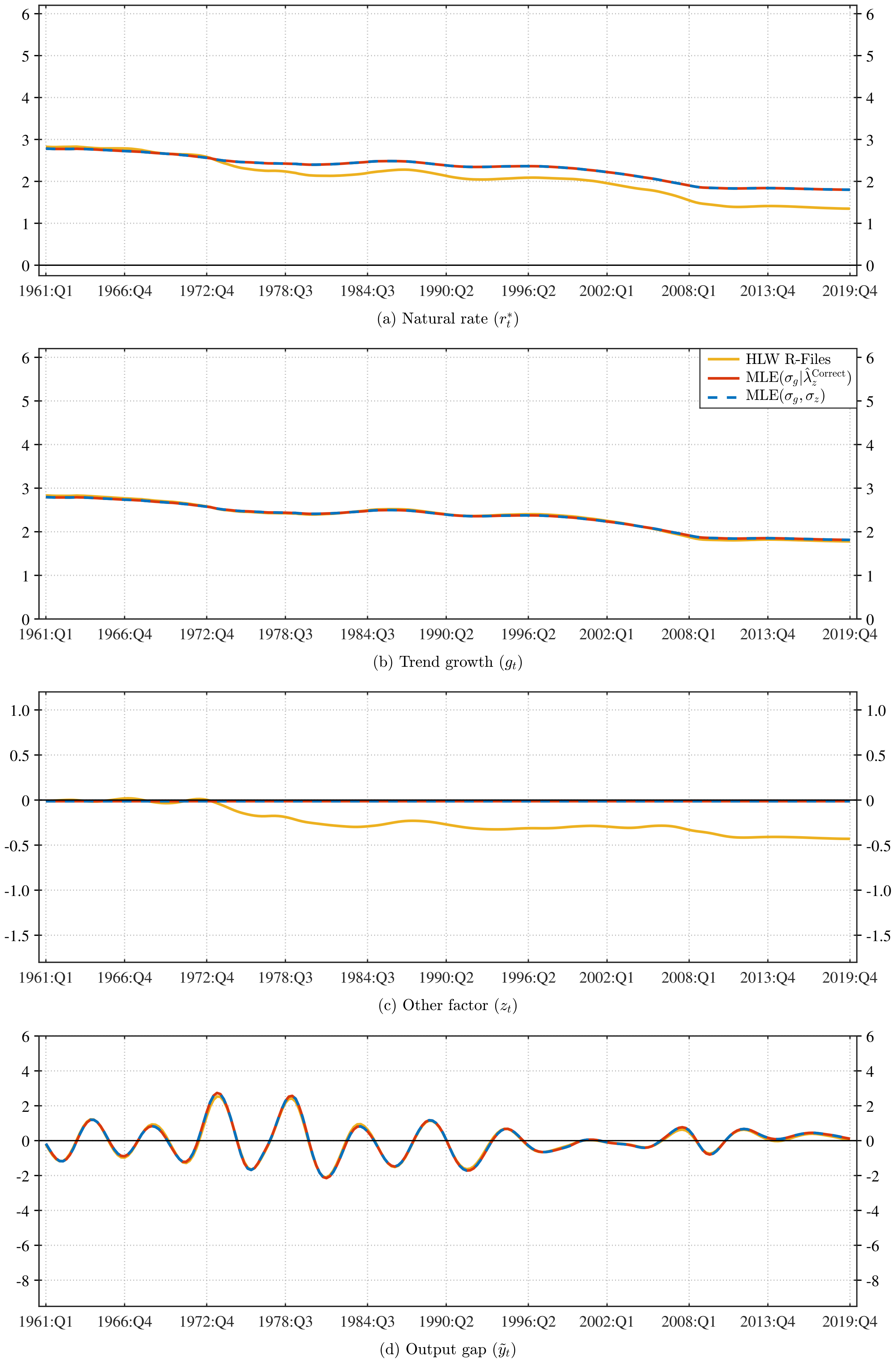
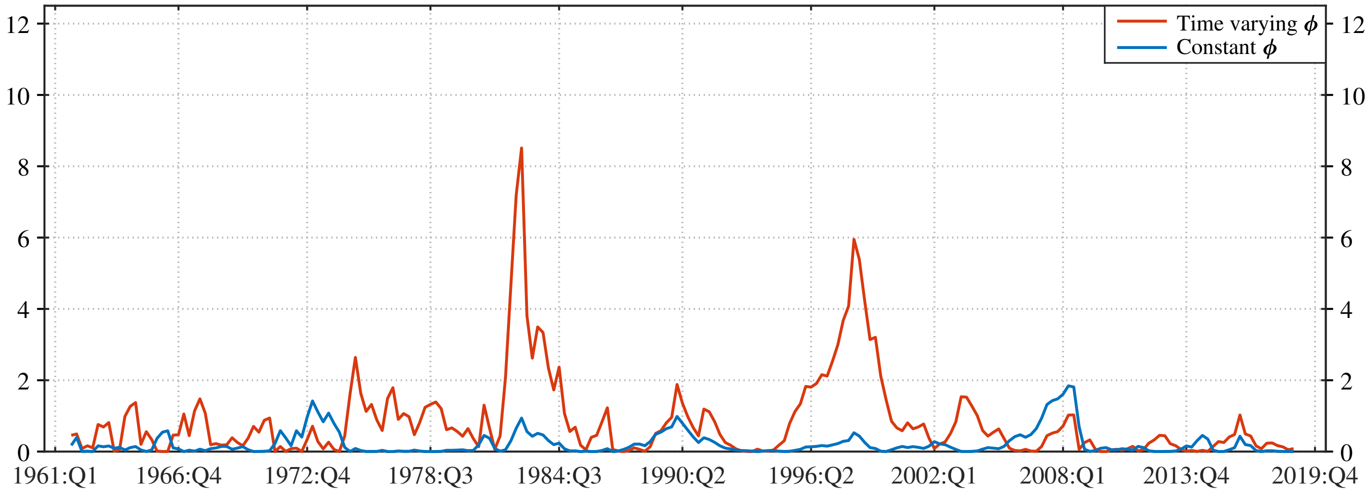


Figure 9: Smoothed estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for the UK

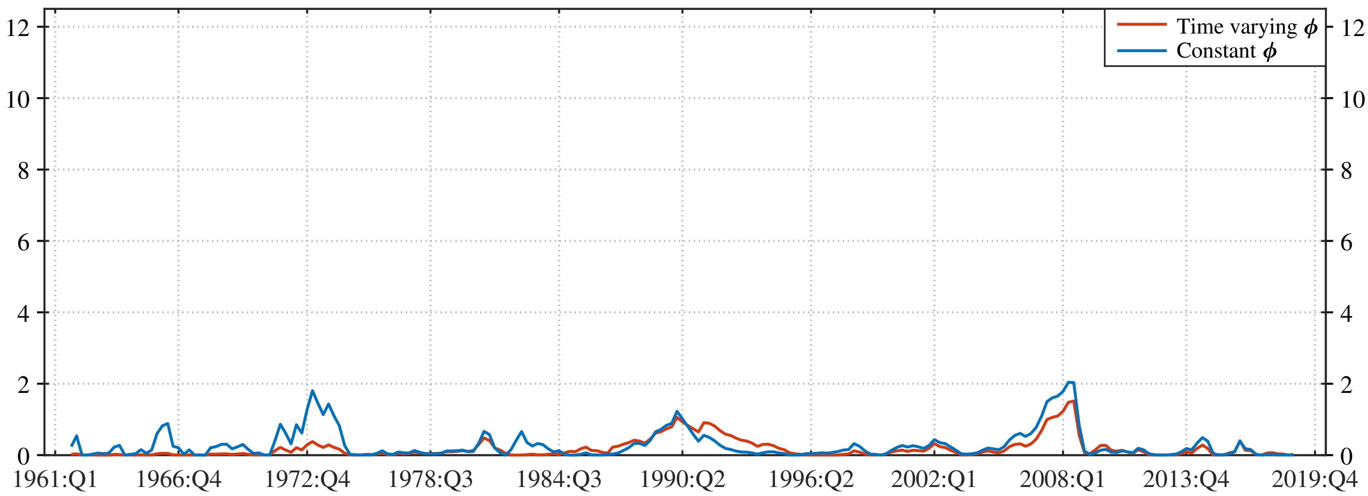
Table 10: Stage 2 parameter estimates for Canada

θ_2	HLW.R-File	HLW($\hat{\sigma}_g^{\text{MLE}}$)	Correct
$a_{y,1}$	1.51416741	1.50661160	1.50350142
$a_{y,2}$	−0.56685374	−0.56099274	−0.55721565
a_r	−0.06316858	−0.06320756	−0.06464243
a_0	−0.09530898	−0.08051872	—
a_g	0.37648328	0.35430677	—
b_π	0.49581071	0.49508617	0.49588052
b_y	0.06028640	0.06298159	0.06225916
$\sigma_{\tilde{y}}$	0.39785436	0.40457203	0.40900866
σ_π	1.38008373	1.37954978	1.38020485
σ_{y^*}	0.58052963	0.57458704	0.57205197
σ_g (implied)	(0.02978747)	0.03253350	0.03365830
λ_g (implied)	0.05131085	(0.05662066)	(0.05883783)
Log-likelihood	−679.90156186	−679.87545118	−679.94373632

Notes: This table reports parameter estimates of the Stage 2 model. The first column ('HLW.R-File') lists the estimates obtained from Holston *et al.*'s (2017) R-Files, where λ_g is fixed at the first stage estimate and σ_g is implied from the Stage 1 signal-to-noise ratio $\lambda_g = \sigma_g/\sigma_{y^*}$. The second column ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') shows estimates when σ_g is computed together with the other parameters of Holston *et al.*'s (2017) Stage 2 model by MLE. The last column ('Correct') provides estimates of the *"correctly specified"* Stage 2 model defined in the left column block of (6), where σ_g is again estimated directly by MLE. Values in round brackets give the implied values of σ_g or λ_g from the $\lambda_g = \sigma_g/\sigma_{y^*}$ relation when either λ_g or σ_g is estimated.



(a) Misspecified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)r_t - a_g g_{t-1} - a_0$



(b) Correctly specified: $GY_t = a_y(L)\tilde{y}_t - a_r(L)[r_t - 4g_t]$

Figure 10: Sequence of F statistics from the correctly and incorrectly specified Stage 2 models for Canada.

Table 11: Stage 2 MUE results of λ_z with corresponding structural break test statistics for Canada

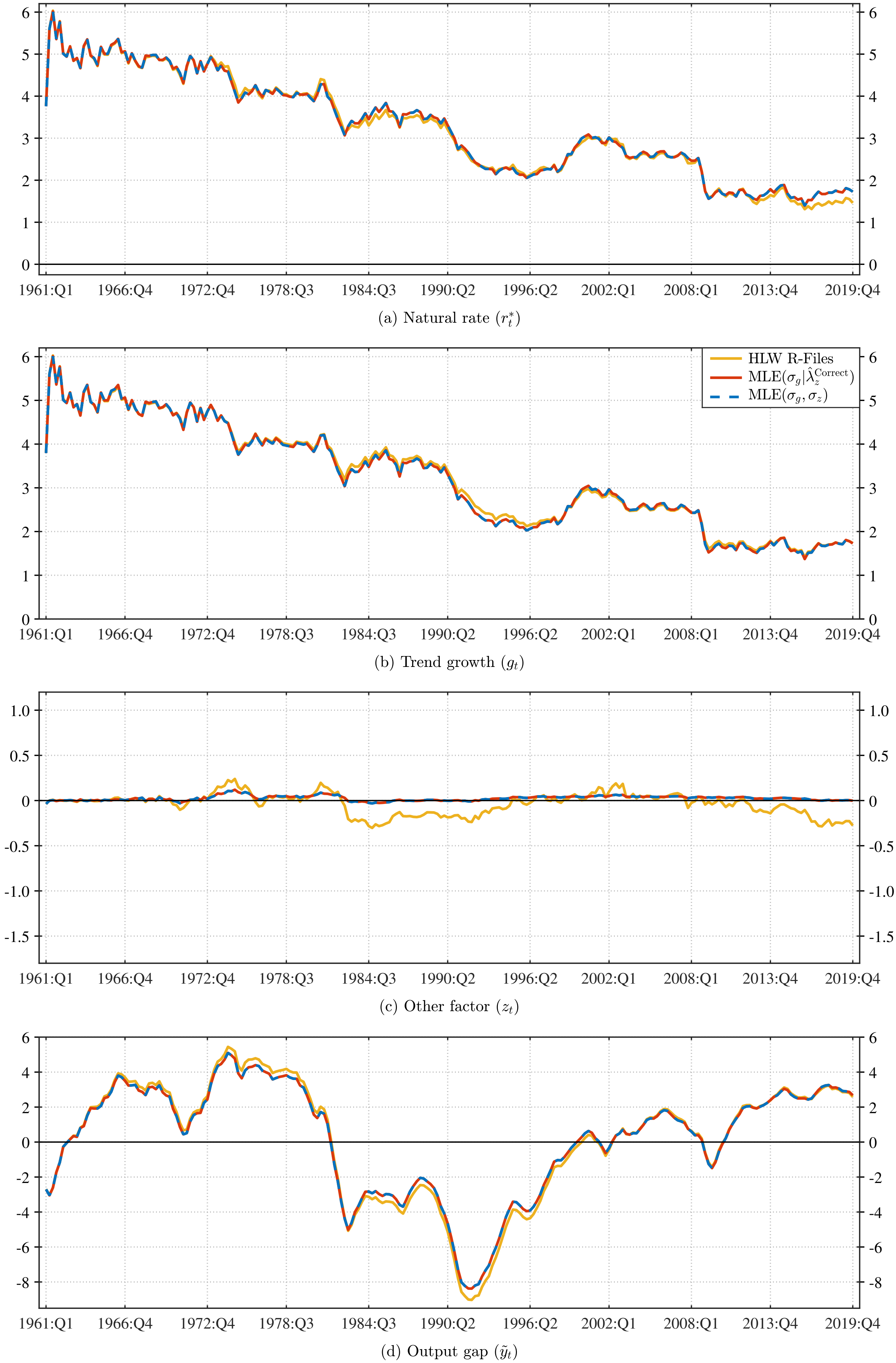
λ_z	Time varying $\boldsymbol{\phi}$					Constant $\boldsymbol{\phi}$					
	HLW.R-File	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]	HLW	[90% CI]	HLW($\hat{\sigma}_g^{\text{MLE}}$)	[90% CI]	Correct	[90% CI]
L	—	0.000000	[0, 0.00]	0.000000	[0, 0.01]	0.000000	[0, 0.00]	0.000000	[0, 0.00]	0.000000	[0, 0.01]
MW	0.009032	0.008855	[0, 0.06]	0.000000	[0, 0.00]	0.000000	[0, 0.01]	0.000000	[0, 0.00]	0.000000	[0, 0.02]
EW	0.016314	0.015288	[0, 0.07]	0.000000	[0, 0.00]	0.000000	[0, 0.01]	0.000000	[0, 0.01]	0.000000	[0, 0.02]
QLR	0.032111	0.031447	[0, 0.11]	0.000000	[0, 0.01]	0.000000	[0, 0.02]	0.000000	[0, 0.02]	0.000000	[0, 0.03]
Corresponding structural break test statistics ($p-$values in parenthesis)											
L	—	0.037529	(0.9450)	0.048128	(0.8850)	0.038851	(0.9350)	0.037529	(0.9450)	0.048128	(0.8850)
MW	0.833493	0.824761	(0.4200)	0.182530	(0.9550)	0.221949	(0.9200)	0.214878	(0.9250)	0.278728	(0.8650)
EW	0.801254	0.761332	(0.2900)	0.101482	(0.9500)	0.126872	(0.9100)	0.122280	(0.9200)	0.161823	(0.8550)
QLR	8.513039	8.272699	(0.0550)	1.514399	(0.8850)	1.842522	(0.8050)	1.793354	(0.8200)	2.040638	(0.7600)

Notes: This table reports the Stage 2 MUE of λ_z obtained from the *"misspecified"* and *"correctly specified"* Stage 2 models. The table is split into left and right column blocks corresponding to the two different structural break test implementations, which are denoted by 'Time varying $\boldsymbol{\phi}$' and 'Constant $\boldsymbol{\phi}$', respectively. The bottom half of the table lists the corresponding structural break test statistics. The results under ('HLW.R-File') report λ_z estimates obtained from Holston *et al.*'s (2017) R-Files for the *"misspecified"* Stage 2 model. The ('HLW($\hat{\sigma}_g^{\text{MLE}}$)') column lists estimates when σ_g is computed by MLE rather than from the first Stage λ_g. Results under the heading ('Correct') are for the *"correctly specified"* Stage 2 model where σ_g is again estimated directly by MLE. The table lists results for all four structural break tests, namely, Nyblom's (1989) L, MW, EW and QLR structural break tests. Values in square brackets in the top part of the table are 90% lower and upper confidence intervals (CIs). Values in parenthesis in the bottom part are $p-$values corresponding to the structural break tests. Both, the CIs as well as the $p-$values, were obtained from Stock and Watson's (1998) GAUSS files.

Table 12: Stage 3 parameter estimates for Canada

θ_3	HLW.R-File	MLE($\sigma_g \vert \hat{\lambda}_z^{\text{Correct}}$)	MLE(σ_g, σ_z)
$a_{y,1}$	1.51607878	1.50339830	1.50339828
$a_{y,2}$	−0.56686382	−0.55701527	−0.55701531
a_r	−0.06620560	−0.06454512	−0.06454510
b_π	0.49906078	0.49635743	0.49635740
b_y	0.05221334	0.06093604	0.06093616
$\sigma_{\tilde{y}}$	0.39354845	0.40917902	0.40917902
σ_π	1.38253182	1.38055866	1.38055865
σ_{y^*}	0.58417740	0.57200959	0.57200958
σ_g (implied)	(0.02997464)	0.03366297	0.03366298
σ_z (implied)	(0.09697384)	(0.00000000)	0.00000001
λ_g (implied)	0.05131085	(0.05885035)	(0.05885036)
λ_z (implied)	0.01631365	0.00000000	(0.00000000)
Log-likelihood	−680.25212417	−680.00345718	−680.00345718

Notes: This table reports the Stage 3 estimates. The first column ('HLW.R-File') gives the estimates from Holston *et al.*'s (2017) R-Files. The second column ('MLE($\sigma_g|\hat{\lambda}_z^{\text{Correct}}$)') shows estimates from the *"correctly specified"* Stage 2 model's MUE of λ_z (based on the EW structural break test), where σ_g is estimated by MLE. The last column ('MLE(σ_g, σ_g)') reports estimates where all parameters, including (σ_g, σ_g), are computed by MLE. Values in round brackets give the implied (σ_g, σ_z) or (λ_g, λ_z) values constructed from the signal-to-noise ratios $\lambda_g = \sigma_g/\sigma_{y^*}$ and $\lambda_z = a_r\sigma_z/\sigma_{\tilde{y}}$.



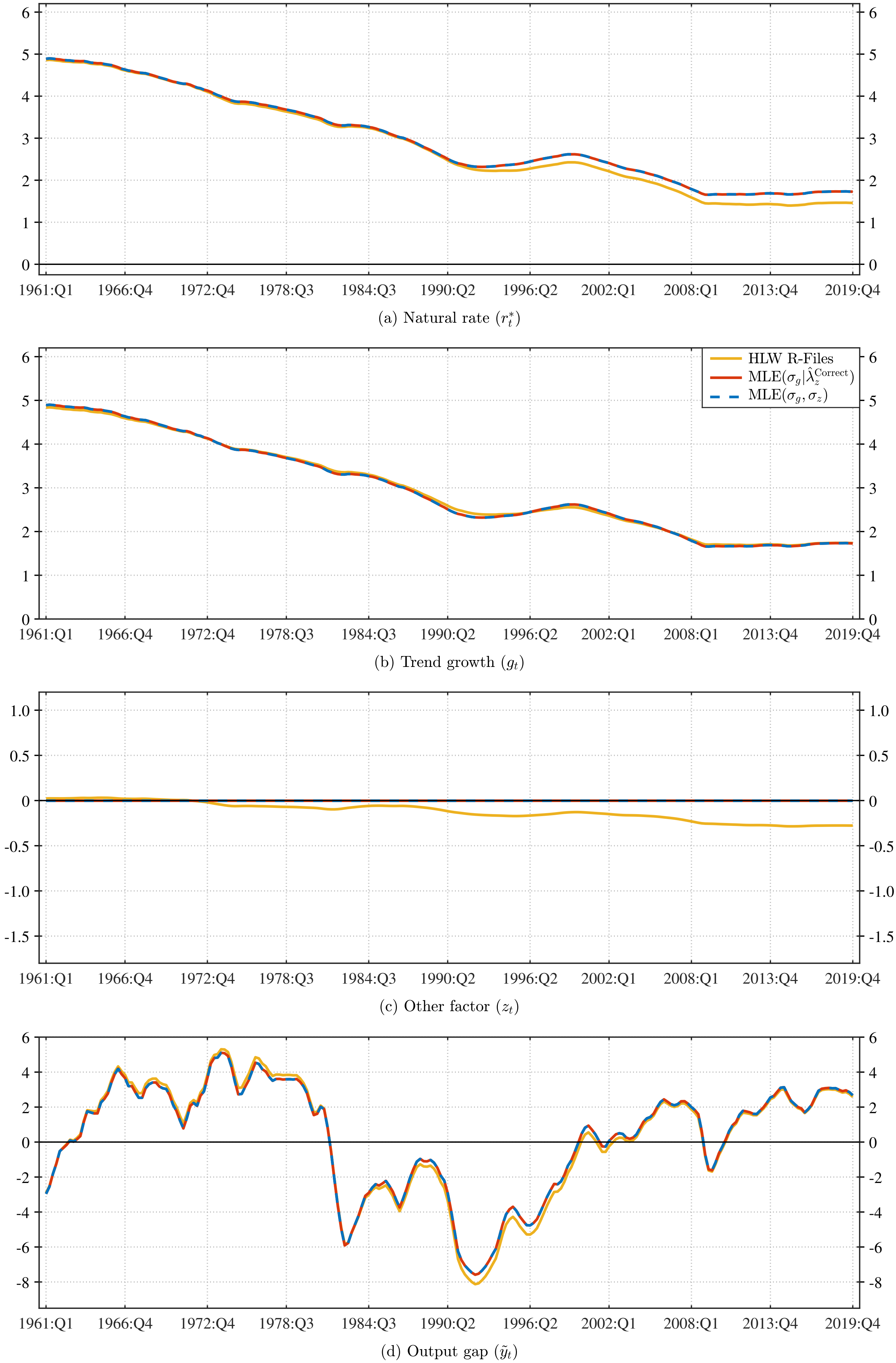
(a) Natural rate (r_t^*)

(b) Trend growth (g_t)

(c) Other factor (z_t)

(d) Output gap ($\tilde{y}_t$)

Figure 11: Filtered estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for Canada



(a) Natural rate (r_t^*)

(b) Trend growth (g_t)

(c) Other factor (z_t)

(d) Output gap ($\tilde{y}_t$)

Figure 12: Smoothed estimates of the natural rate r_t^*, annualized trend growth g_t, *'other factor'* z_t, and the output gap (cycle) variable $\tilde{y}_t$ for Canada